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Algebraic Topology

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Contents

1 Point–Set Topology

1.1 Topological Spaces & Continuous Maps

Definition 1.1. Let X be a set. A *topology* on X is a system $\mathscr{T} \subset \mathscr{P}(X)$ closed under arbitrary union and finite intersection. Subsets $U \subset X$ which are elements of $\mathscr{T}$ are called *open*. The pair $(X, \mathscr{T})$ is called a *topological space*.

Examples 1.2.

(i) For any set X, the system $\mathscr{T} = \mathscr{P}(X)$ is a topology—the *discrete* topology.

(ii) Similarly, for any set X, the system $\mathscr{T} = \{\emptyset, X\}$ is a topology—the *indiscrete* topology.

(iii) Let (X, d) be a metric space. Define $\mathscr{T} \subset \mathscr{P}(X)$ such that $U \in \mathscr{T}$ if and only if for any $x \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset U$. This defines a topology on X—the topology *induced* by the metric d.

Definition 1.3. Let $(X, \mathscr{T}_X)$ be a topological space and consider any subset $A \subset X$. The topology on A defined by

$$\mathscr{T}_A = \{U \cap A \colon U \in \mathscr{T}_X\}$$

is called the *subspace* topology. Subsets of topological spaces will always carry the subspace topology unless indicated otherwise.

Definition 1.4. A map $f \colon X \longrightarrow Y$ between topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ is called *continuous* if, for any open set $U \subset Y$, the inverse image $f^{-1}(U) \subset X$ is open.

Remark 1.5. Continuity is stable under composition of maps, that is if $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$ are continuous, then so is $g \circ f \colon X \longrightarrow Z$. Hence, we arrive at a *category* **Top** of topological spaces whose morphisms are continuous maps.

Definition 1.6. Let $(X, \mathscr{T})$ be a topological space and fix $a \in X$. A subset $U \subset X$ is called a *neighbourhood* of a if there exists some open set $V \subset X$ such that $a \in V$ and $V \subset U$.

Note that neighbourhoods need not be open! The concept of neighbourhoods allows us to define convergence of sequences and continuity at a point as follows.

Definition 1.7. Let $(X, \mathscr{T})$ be a topological space. A sequence $(x_n)_{n\in\mathbb{N}}$ in X is *convergent* with limit $a \in X$ if (x_n) is eventually contained in every neighbourhood of a, i. e. for every neighbourhood U of a there is an $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$.

Remark 1.8. If (X, d) is a metric space and $\mathscr{T}$ is the topology induced by d, then convergence in $(X, \mathscr{T})$ recovers convergence in the metric space (X, d).

Definition 1.9. A map $f \colon X \longrightarrow Y$ between topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ is called *continuous at* $a \in X$ if for every neighbourhood V of $f(a)$ there is a neighbourhood U of a such that $f(U) \subset V$.

Remark 1.10.

(i) Again, if (X, d_X) and (Y, d_Y) are metric spaces and $\mathscr{T}_X$ and $\mathscr{T}_Y$ are the induced topologies, then continuity at a point $a \in X$ of a map $f \colon X \longrightarrow Y$ in this sense recovers the usual definition of continuity at a in metric spaces.

(ii) Let $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ be topological spaces. Then a map $f \colon X \longrightarrow Y$ is continuous if and only if it is continuous at every point $a \in X$.

Definition 1.11. A subset $A \subset X$ in a topological space $(X, \mathscr{T}_X)$ is called *closed* if $X \setminus A$ is open.

Remark 1.12. The family of closed sets in a topological space $(X, \mathscr{T}_X)$ is closed under arbitrary intersection and finite union. This is immediate from the definition of a topology on X by passing to the complement.

Definition 1.13. Let $(X, \mathscr{T})$ be a topological space and $A \subset X$ a subset. The *closure* $\overline{A}$ of A is defined by

$$\overline{A} = \bigcap \{B \subset X \colon B \supset A \text{ and } B \text{ is closed}\}.$$

Analogously, the *interior* A° of A is defined by

$$A^\circ = \bigcup \{U \subset X \colon U \subset A \text{ and } U \text{ is open}\}.$$

The *boundary* ∂A of A is defined as $\partial A = \overline{A} \setminus A^\circ$.

It is immediate that, in the situation above, $\overline{A}$ and ∂A are closed subsets of X and A° is an open subset of X.

Definition 1.14. A continuous map $f \colon X \longrightarrow Y$ between topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ is called a *homeomorphism* if it admits a continuous inverse, i. e. if there exists a continuous map $g \colon Y \longrightarrow X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. If there exists a homeomorphism $X \longrightarrow Y$, then X and Y are called *homeomorphic*.

Remark 1.15.
- (i) Not every continuous bijection is a homeomorphism!
- (ii) Let $(X, \mathscr{T})$ be a topological space and $A \subset X$ a subset. Endow A with the subspace topology with respect to X and consider the inclusion $i \colon A \hookrightarrow X$. Then i induces a homeomorphism $i \colon A \xrightarrow{\sim} i(A)$, i. e. i is an *embedding*.
- (iii) A map $f \colon X \longrightarrow Y$ between topological spaces is a homeomorphism if and only if f is bijective, continuous and *open*, that is $f(U)$ is open for all open sets $U \subset X$.

1.2 Separation Properties & Compactness

Definition 1.16. A topological space $(X, \mathscr{T})$ is called *Hausdorff* if distinct points can be separated by disjoint neighbourhoods. Formally, for all $x, y \in X$ with $x \neq y$ there exist neighbourhoods U and V of x and y respectively such that $U \cap V = \emptyset$.

For example, if (X, d) is a metric space, the topology $\mathscr{T}$ induced by d makes $(X, \mathscr{T})$ into a Hausdorff space.

Remark 1.17. If $(X, \mathscr{T})$ is a Hausdorff space, the following properties hold:
- (i) Points are closed: for every $x \in X$ the set $\{x\} \subset X$ is closed.
- (ii) Limits of convergent sequences are unique.

Definition 1.18. A topological space $(X, \mathscr{T})$ is called *compact* if every open cover has a finite subcover: If $(U_i)_{i \in I}$ is a family of open subsets of X such that $X = \bigcup_{i \in I} U_i$, then there exists a finite set $J \subset I$ such that $X = \bigcup_{j \in J} U_j$.

Recall the Heine–Borel theorem: A subset $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded. We will now give some consequences of compactness.

Proposition 1.19. *If $(X, \mathscr{T})$ is a compact topological space and $A \subset X$ is a closed subset, then A, endowed with the subspace topology, is compact.*

Proof. Let $(U_i)_{i \in I}$ be an open cover of A. Choose open subsets $U'_i \subset X$ such that $U_i = U'_i \cap A$ and observe that $X \smallsetminus A, (U'_i)_{i \in I}$ is an open cover of X. Because X is compact there exists a finite set $J \subset I$ such that $X = (X \smallsetminus A) \cup \bigcup_{j \in J} U'_j$. But then $A = \bigcup_{j \in J} U_j$, i. e. $(U_j)_{j \in J}$ is a finite subcover of $(U_i)_{i \in I}$. □

Proposition 1.20. *Let $f \colon X \longrightarrow Y$ be a continuous map between topological spaces $(X, \mathscr{T}_X)$ and $(Y, \mathscr{T}_Y)$ and assume $(X, \mathscr{T}_X)$ to be compact. Then the image $f(X) \subset Y$ is compact in the subspace topology.*

Proof. First observe that the map $f \colon X \longrightarrow f(X)$ induced by f is continuous. Any open cover $(U_i)_{i \in I}$ of $f(X)$ pulls back to an open cover $(f^{-1}(U_i))_{i \in I}$ of X. Hence, there exists a finite set $J \subset I$ such that $(f^{-1}(U_j))_{j \in J}$ is a cover of X. But then $\bigcup_{j \in J} U_j = f(X)$. □

Proposition 1.21. *If $(X, \mathscr{T})$ is a Hausdorff space and $A \subset X$ is compact in the subspace topology, then A is closed in X.*

Proof. We will show that $X \smallsetminus A$ is open. For any point $x \in X \smallsetminus A$ there exists an open neighbourhood U of x with $U \cap A = \emptyset$: For any $y \in A$ there exist disjoint open neighbourhoods U_y of x and V_y of y. Then $A = A \cap \bigcup_{y \in A} V_y$, hence by compactness there exists a finite subset $A' \subset A$ such that $A = A \cap \bigcup_{y \in A'} V_y$. Then $U = \bigcap_{y \in A'} U_y$ is an open neighbourhood of x with $U \cap A = \emptyset$.

Now, choosing an open neighbourhood U_x as above for any point $x \in X \smallsetminus A$, it is immediate that $\bigcup_{x \in X \smallsetminus A} U_x = X \smallsetminus A$ is open. □

Definition 1.22. A subset A of a topological space $(X, \mathscr{T})$ is called *relatively compact* if $\overline{A}$ is compact.

For example, if $A \subset \mathbb{R}^n$ is bounded, then by Heine–Borel A is relatively compact. For metric spaces there is a related notion:

Definition 1.23. A metric space (X, d) is called *precompact* or *totally bounded* if for any $\varepsilon > 0$ there exists a finite *ε–net* in X, that is a finite collection of points $(x_i)_{i \in I}$ in X such that $X = \bigcup_{i \in I} B_\varepsilon(x_i)$.

Theorem 1.24. *Let (X, d) be a metric space. Then the following are equivalent:*
(i) *X is compact.*
(ii) *Any sequence (x_n) in X has a convergent subsequence.*
(iii) *X is precompact and complete.*

Proof. For (i→ii) let (x_n) be a sequence in X. Consider the sets

$$X_n = \overline{\{x_n, x_{n+1}, \dots\}}.$$

Clearly $X_n \supset X_{n+1}$ for all n. Since X is compact $X_\infty = \bigcap_{n \in \mathbb{N}} X_n$ is not empty: Otherwise we would have $X = \bigcup_{n \in \mathbb{N}} (X \smallsetminus X_n)$. Choosing a finite set $I \subset \mathbb{N}$ with $X = \bigcup_{n \in I} (X \smallsetminus X_n)$ one sees

$\emptyset = \bigcap_{n\in I} X_n = X_{\max I}$. But this is impossible. Hence, we can choose some $x_\infty \in X_\infty$. For $k \in \mathbb{N}$ pick $n_k \in \mathbb{N}$ such that $x_{n_k} \in B_{1/k}(x_\infty)$. Choosing the n_k to be distinct, $(x_{n_k})_k$ is a subsequence converging to x_∞.

The completeness part of (ii→iii) is clear. To see that X has to be precompact, suppose there exists some $\varepsilon > 0$ such that X admits no finite ε–net. Then for any finite collection $x_1, \dots, x_n$ of points of X there exists some $x_{n+1} \in X$ with $d(x_{n+1}, x_i) \geq \varepsilon$ for all $1 \leq i \leq n$. Inductively, one obtains a sequence (x_n) in X that admits no convergent subsequence which contradicts (ii).

Now suppose that X is precompact and complete and consider any open cover $(U_i)_{i\in I}$ of X. For contradiction, suppose that $(U_i)_{i\in I}$ has no finite subcover. Because X is assumed precompact, it admits a finite 1–net, e. g. $X = \bigcup_{j=1}^{N(1)} B_1(y_j)$. Then there exists some $j_0 \in \{1, \dots, N(1)\}$ such that $B_1(y_{j_0})$ is not covered by finitely many of the U_i. Set $x_0 = y_{j_0}$ and $B_0 = B_1(x_0)$.

Continuing inductively one obtains a sequence of points (x_n) of X such that $B_n = B_{2^{-n}}(x_n)$ cannot be covered by finitely many of the U_i and furthermore this sequence may be chosen such that $B_{n-1} \cap B_n \neq \emptyset$ for all n, for otherwise B_{n-1} could be covered by finitely many of the U_i. Then $d(x_n, x_{n+1}) \leq 2^{-n+1}$ and indeed (x_n) is a Cauchy sequence: For $m > n$ one has

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \leq 2^{-n+1} + \cdots + 2^{-m+2} = \\ &= 2^{-n+1}(1 + \cdots + 2^{n-m+1}) \leq 2^{-n+2}. \end{aligned}$$

Completeness gives the existence of a limit x_∞ for (x_n). Take some $i_0 \in I$ such that $x_\infty \in U_{i_0}$. Then there exists some $N \in \mathbb{N}$ such that $B_N \subset U_{i_0}$, i. e. B_N can be covered by finitely many of the U_i which is impossible by the construction of (x_n). □

1.3 Connectedness & Path–Connectedness

From now on we will write X for the topological space $(X, \mathscr{T})$ if the topology is understood.

Definition 1.25. A topological space X is called *connected*, if it admits no partition $X = U \cup V$ with $U, V \subset X$ open, nonempty and disjoint.

Proposition 1.26. *Let $f\colon X \longrightarrow Y$ be a continuous map. If X is connected, then so is $f(X)$.*

Proof. Any partition of $f(X)$ pulls back to a partition of X. □

Proposition 1.27. *Let $(A_j)_{j\in J}$ be a family of connected subsets of X such that $\bigcap_{j\in J} A_j \neq \emptyset$. Then $\bigcup_{j\in J} A_j$ is connected.*

Proof. Write $A = \bigcup_{j\in J} A_j$ and suppose there exists a nontrivial partition $A = U \cup V$ with $U, V \subset A$ open. Since the A_j are connected, for any j, either $A_j \cap U = \emptyset$ or $A_j \cap V = \emptyset$. Fix $j_0 \in J$ and suppose without loss of generality that $A_{j_0} \cap V = \emptyset$. Then, because $\bigcap_{j\in J} A_j \neq \emptyset$, for any $j \in J$ one has $A_j \cap U \neq \emptyset$. Hence, $A \subset U$. But this contradicts $V \neq \emptyset$. □

Proposition 1.28. *Let $A \subset X$ be some subspace. If A is connected, then so is $\overline{A}$.*

Proof. Suppose there exists a nontrivial open partition $\overline{A} = U \cup V$. Write $U = U' \cap \overline{A}$ and $V = V' \cap \overline{A}$ with open subset $U', V' \subset X$. Then $A = (U' \cap A) \cup (V' \cap A)$ and of course $U' \cap V' \cap A = \emptyset$. Because A is connected, it is contained in U' or V', for instance $A \subset U'$. Then $A \subset X \setminus V'$, so $\overline{A} \subset X \setminus V'$. Hence, $\overline{A} \subset U$ in contradiction to $V \neq \emptyset$. □

Definition 1.29. Let $x_0 \in X$. The *connected component* of x_0 is defined as

$$X_0 = \bigcup \{A \subset X \colon x_0 \in A \text{ and } A \text{ is connected}\}.$$

Remark 1.30.

(i) Of course, X is the disjoint union of its connected components.
(ii) The connected components are connected and closed.

Definition 1.31. Denote by $I = [0,1]$ the unit interval. A *path* in X is a continuous map $\gamma \colon I \longrightarrow X$. A path γ is called *closed* if $\gamma(0) = \gamma(1)$. In this case, we say that γ is based at $\gamma(0) = \gamma(1)$.

For any point $x \in X$, we denote by c_x the *constant path* at x; given any path $\gamma \colon I \longrightarrow X$, we define its *inverse path* via $\gamma^-(t) = \gamma(1-t)$. The *concatenation* of paths $\gamma_1, \gamma_2 \colon I \longrightarrow X$ is defined by

$$\gamma_1 * \gamma_2(t) = \begin{cases} \gamma_1(2t) & t \in [0, 1/2] \\ \gamma_2(2t-1) & t \in [1/2, 1]. \end{cases}$$

Define an equivalence relation on X such that $x \sim y$ if and only if there exists a path $\gamma \colon I \longrightarrow X$ with $\gamma(0) = x$ and $\gamma(1) = y$. The *path components* of X are the equivalence classes of $\sim$. A space X is called *path–connected* if it has precisely one path component.

Remark 1.32.

(i) Again, X is the disjoint union of its path components.
(ii) A space X is path connected if and only any two points of X can be joined by a path.
(iii) If X is path connected, then it is connected. Indeed, if there were a nontrivial partition $X = U \cup V$, choose $x \in U$ and $y \in V$ and a path $\gamma \colon I \longrightarrow X$ joining x and y. Then $\gamma(I)$ is connected, but the partition $X = U \cup V$ would restrict to a nontrivial partition of $\gamma(I)$.
(iv) The condition of path–connectedness is strictly stronger than connectedness: There exist connected spaces which are not path–connected.

Definition 1.33. A topological space X is called *locally (path–)connected* if any point $x \in X$ admits a neighbourhood basis of (path–)connected subsets of X, i. e. for any neighbourhood U of x there exists a (path–)connected neighbourhood $V \subset U$ of x.

For example, euclidean space $\mathbb{R}^n$ is locally path–connected, because balls are path–connected and give a neighbourhood base for any point.

Remark 1.34. There do exist path–connected spaces which are not even locally connected! An example can be constructed by considering the set $N = \{0\} \cup \{(1/n, 0) \colon n \in \mathbb{N}\} \subset \mathbb{R}^2$ and taking lines from $(0,1)$ to all points of N. In the resulting space, the point $(0,0)$ admits no small connected neighbourhood.

Proposition 1.35. *If X is locally (path–)connected, then its connected components (path–components respectively) are open.*

Proof. Pick $x_0 \in X$ and denote by X_0 its connected component (or path–component). If $y \in X_0$ and U is a (path–)connected neighbourhood of y, then $U \subset X_0$. This implies the claim. □

Proposition 1.36. *Let X be locally path–connected. Then an open subset $U \subset X$ is path–connected if and only if U is connected.*

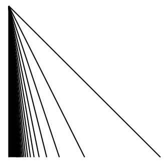


Figure 1: The space of Remark 1.34.

Proof. For the nontrivial direction let $U \subset X$ be a connected open subset. Because U is open, it is locally path–connected. Hence, its path–components are open and closed and, if there were more than one, this would yield a nontrivial partition of U. $\square$

Remark 1.37. The previous proposition implies that the connected components of a locally path–connected space are open and coincide with the path–components.

1.4 Products, Sums, Quotients

We already know one construction on topological spaces: To any topological space $(X, \mathscr{T}_X)$ with a subset $A \subset X$ we can associate the topological space $(A, \mathscr{T}_A)$, that is A with the subspace topology. This construction comes with a natural continuous map $i\colon A \hookrightarrow X$.

Definition 1.38. Let X be a set and consider topologies $\mathscr{T}$ and $\mathscr{T}'$ on X. The topology $\mathscr{T}$ is called *finer* than $\mathscr{T}'$ (or equivalently $\mathscr{T}'$ is *coarser* than $\mathscr{T}$), if $\mathscr{T}' \subset \mathscr{T}$. That is, any open set with respect to $\mathscr{T}'$ is also open with respect to $\mathscr{T}$.

Remark 1.39. Every topology is coarser than the discrete topology and finer than the indiscrete topology. This corresponds to the trivial fact that $\{\emptyset, X\} \subset \mathscr{T} \subset \mathscr{P}(X)$.

Now we can formulate an alternative characterisation of the subspace topology. Namely, $\mathscr{T}_A$ is the coarsest topology on A such that $i\colon A \hookrightarrow X$ is continuous. Furthermore, the subspace topology is characterised by the following *universal property*: For any topological space Y, a map $f\colon Y \longrightarrow A$ is continuous if and only if $i \circ f\colon Y \longrightarrow X$ is continuous:

$$\begin{array}{ccc} A & \xrightarrow{\;i\;} & X. \\ {\scriptstyle f}\big\uparrow & \nearrow_{i\circ f} & \\ Y & & \end{array}$$

The following is immediate.

Proposition 1.40. *Let X be a set and $\mathscr{S} \subset \mathscr{P}(X)$ an arbitrary subset. Then there exists a unique coarsest topology $\mathscr{T}(\mathscr{S})$ containing $\mathscr{S}$.*

Definition 1.41. For a set X and an arbitrary subset $\mathscr{S} \subset \mathscr{P}(X)$, the topology $\mathscr{T}(\mathscr{S})$ of Proposition 1.40 is called the topology *generated* by $\mathscr{S}$.

Definition 1.42. Let $(X, \mathscr{T})$ be a topological space. A subsystem

(i) $\mathscr{B} \subset \mathscr{T}$ is called *basis* of $\mathscr{T}$ if any $U \subset \mathscr{T}$ can be written as a union of set in $\mathscr{B}$.

(ii) $\mathscr{S} \subset \mathscr{T}$ is called *subbasis* if the system of finite intersection of sets in $\mathscr{S}$ is a basis for $\mathscr{T}$.

Remark 1.43.

(i) If $\mathscr{T}$ is a topology on X and $\mathscr{S} \subset \mathscr{T}$ is some subsystem, then $\mathscr{S}$ is a subbasis if and only if $\mathscr{T}(\mathscr{S}) = \mathscr{T}$.

(ii) A map $f\colon (X, \mathscr{T}_X) \longrightarrow (Y, \mathscr{T}_Y)$ is continuous if and only if $f^{-1}(S) \in \mathscr{T}_X$ for all $S \in \mathscr{S}_Y$, where $\mathscr{S}_Y$ is a subbasis for $\mathscr{T}_Y$.

Let $(X_i, \mathscr{T}_i)$ be a family of topological spaces indexed by an arbitrary set I. Consider the cartesian product $X = \prod_{i \in I} X_i$ together with the projections $\pi_i\colon X \longrightarrow X_i$.

Definition 1.44. The *product topology* $\mathscr{T}$ on X is the topology generated by the cylinders $\pi_i^{-1}(U_i)$ for $U_i \in \mathscr{T}_i$.

Proposition 1.45.

(i) *The topology $\mathscr{T}$ of Definition 1.44 is the coarsest topology on X such that all projections $\pi_i\colon X \longrightarrow X_i$ are continuous.*

(ii) *A map $f\colon Y \longrightarrow X$ from a topological space Y is continuous if and only if all $\pi_i \circ f$ are continuous:*

$$\begin{array}{ccc} X & \xrightarrow{\pi_i} & X_i. \\ {\scriptstyle f}\Big| & \nearrow{\scriptstyle \pi_i \circ f} & \\ Y & & \end{array}$$

Proof. The continuity of π_i requires $\pi_i^{-1}(U_i)$ to be open for every open subset $U_i \subset X_i$, which proves (i). Furthermore, $f^{-1}(\pi_i^{-1}(U_i)) = (\pi_i \circ f)^{-1}(U_i)$ is open for every open set $U_i \subset X$. This proves continuity of f. ☐

Remark 1.46. If X and Y are Hausdorff, then so is $X \times Y$.

Proposition 1.47. *If X and Y are compact, then so is $X \times Y$.*

Proof. Let $(U_i)_{i \in I}$ be an open cover of $X \times Y$. Without loss of generality we may assume that $U_i = V_i \times W_i$ for open subsets $V_i \subset X$ and $W_i \subset Y$. Fix some $x \in X$. Then $Y \cong \{x\} \times Y$ is compact, so there is a finite subset $J(x) \subset I$ such that $\{x\} \times Y \subset \bigcup_{j \in J(x)} V_j \times W_j$. Now, the set $V_x = \bigcap_{j \in J(x)} V_j$ is open in X and the family $(V_x)_{x \in X}$ is an open cover of X. Because X is compact, there is some finite subset $X' \subset X$ such that $X = \bigcup_{x \in X'} V_x$. Then, writing $J' = \bigcup_{x \in X'} J(x)$, the family $(V_j \times W_j)_{j \in J'}$ is a finite subcover of $(U_i)_{i \in I}$. ☐

Remark 1.48. More generally, one has *Tychonoff's theorem*: If $(X_i)_{i \in I}$ is an arbitrary collection of compact spaces, then $\prod_{i \in I} X_i$ is compact.

Let $(X_i, \mathscr{T}_i)$ be topological spaces indexed by some set I. Let $X = \coprod_{i \in I}$ be the disjoint union and denote by $\iota_i\colon X_i \longrightarrow X$ the inclusions.

Definition 1.49. The *sum topology* $\mathscr{T}$ on X is defined such that $U \in \mathscr{T}$ if and only if $U \cap X_i \in \mathscr{T}_i$ for all $i \in I$.

The proof of the following is immediate.

Proposition 1.50.

(i) *The topology $\mathscr{T}$ of Definition 1.49 is the finest topology on X such that all inclusions $\iota_i\colon X_i \longrightarrow X$ are continuous.*

(ii) *A map $f\colon X \longrightarrow Y$ to another topological space Y is continuous if and only if all maps $f \circ \iota_i\colon X_i \longrightarrow Y$ are continuous:*

$$\begin{array}{ccc} X_i & \xrightarrow{\ \iota_i\ } & X. \\ & \underset{f\circ\iota_i}{\searrow} & \downarrow f \\ & & Y \end{array}$$

Remark 1.51.

(i) The subspace topology of $X_i \subset X$ is the original topology $\mathscr{T}_i$.

(ii) Let X be the disjoint union of subsets X_i for $i \in I$. Then X is the sum of the X_i if and only if all $X_i \subset X$ are open.

Let X be a topological space and $\sim$ some equivalence relation on X. We denote by $X/\sim$ the set of equivalence classes and by $\pi\colon X \longrightarrow X/\sim$ the projection.

Definition 1.52. The topology on $X/\sim$ such that $U \subset X/\sim$ is open if and only if $\pi^{-1}(U) \subset X$ is open is called the *quotient topology* on $X/\sim$.

Again, the proof of the following proposition is straightforward.

Proposition 1.53.

(i) *The quotient topology is the finest topology on $X/\sim$ such that $\pi\colon X \longrightarrow X/\sim$ is continuous.*

(ii) *A map $f\colon X/\sim \longrightarrow Y$ to any other space Y is continuous if and only if $f \circ \pi\colon X \longrightarrow Y$ is continuous:*

$$\begin{array}{ccc} X & \xrightarrow{\ \pi\ } & X/\sim \\ & \underset{f\circ\pi}{\searrow} & \downarrow f \\ & & Y \end{array}$$

Remark 1.54.

(i) If X is compact, then so is $X/\sim$.

(ii) In general, $X/\sim$ may fail to be Hausdorff even if X is. For example, consider the relation $\sim$ on $X = \mathbb{R}$ such that $x \sim y$ if and only if $x - y \in \mathbb{Q}$.

Examples 1.55.

(i) Let $A \subset X$ and define an equivalence relation $\sim$ on X such that

$$x \sim y \iff \begin{cases} x, y \in A & \text{or} \\ x = y. \end{cases}$$

Then one may picture $X/\sim$ as X with the subspace A collapsed to a point. We usually write X/A instead of $X/\sim$. For example, consider the closed n–disk $D^n \subset \mathbb{R}^n$ and its boundary, the $(n-1)$–sphere $S^{n-1} = \partial D^n$. Then $D^n/S^{n-1} \cong S^n$, the homeomorphism being induced by the map $f\colon D^n \longrightarrow S^n \subset \mathbb{R}^{n+1}$ such that

$$f(x) = \left(\cos \pi \|x\|, \frac{x \sin \pi \|x\|}{\|x\|} \right).$$

(ii) Let $I^2 \subset \mathbb{R}^2$ be the unit square. Define an equivalence relation $\sim$ on X such that

$$(s_1, s_2) \sim (t_1, t_2) \iff \begin{cases} s_1 = t_1 \text{ and } s_2 = t_2 & \text{or} \\ s_1 = t_1 \text{ and } s_2, t_2 \in \{0,1\} & \text{or} \\ s_2 = t_2 \text{ and } s_1, t_1 \in \{0,1\}. \end{cases}$$

The quotient space $T^2 = I^2/\!\sim$ is called the *2–torus*. It can be shown that $T^2 \cong S^1 \times S^1$. Specifically, a homemorphism is induced by

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow S^1 \times S^1 \\ (\theta, \varphi) &\longmapsto (e^{2\pi i \theta}, e^{2\pi i \varphi}). \end{aligned}$$

Some further quotients of I^2 are pictured in (picture). More generally, one can construct closed, orientable surfaces Σ_g of any genus $g \geq 0$ as a quotient of a $4g$–gon in $\mathbb{R}^2$.

Group actions provide further examples of quotient spaces.

Definition 1.56. Let G be a group and X a topological space. A *(left) action* of G on X is a continuous map

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto gx, \end{aligned}$$

where G is given the discrete topology, satisfying $g(hx) = (gh)x$ and $ex = x$.

Equivalently, a group action is a group homomorphism $G \longrightarrow \mathrm{Aut}(X)$ from G to the group $\mathrm{Aut}(X)$ of homeomorphisms $X \longrightarrow X$. In this description g maps to the homeomorphism $\ell_g\colon X \longrightarrow X$ with $\ell_g(x) = gx$—the *left translation* by g.

Definition 1.57. For $x \in X$ the set $Gx = \{gx\colon g \in G\}$ is called the *orbit* of x. The subgroup $G_x = \{g \in G\colon gx = x\}$ is called the *stabiliser* or *isotropy subgroup* of x. The *orbit space* X/G is the quotient space $X/\!\sim$ for the equivalence relation $\sim$ such that $x \sim y$ if and only if $gx = y$ for some $g \in G$.

The action of G on X is called *transitive* if $X/G = *$. It is called *free* if $G_x = 1$ for all $x \in X$.

Examples 1.58. Real projective space $\mathbb{RP}^n$ is the space of 1–dimensional subspaces of $\mathbb{R}^{n+1}$. More formally, $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\!\sim$ where $\sim$ is given by

$$v \sim w \iff w = \lambda v \text{ for some } \lambda \in \mathbb{R}^\times.$$

The equivalence class of $(x_0, \dots, x_n) = x \in \mathbb{R}^{n+1} \setminus \{0\}$, i. e. the line spanned by x, is denoted by $[x_0 : \dots : x_n]$. The x_i are called the *homogeneous coordinates* of $[x] \in \mathbb{RP}^n$. They are only determined up to multiplication by $\lambda \in \mathbb{R}^\times$.

An alternative characterisation may be obtained from group actions. The multiplicative group $\mathbb{Z}^\times$ acts on S^n by scalar multiplication. The left translation $\ell_{-1}\colon S^n \longrightarrow S^n$ is called the *antipodal* map and of course $\ell_{-1}(x) = -x$. It is a nice exercise that the inclusion $S^n \longrightarrow \mathbb{R}^{n+1} \setminus \{0\}$ induces a homeomorphism $S^n/\mathbb{Z}^\times \xrightarrow{\sim} \mathbb{RP}^n$. In particular, this implies that $\mathbb{RP}^n$ is compact. Furthermore, it may be shown that $\mathbb{RP}^n$ is Hausdorff.

2 Fundamental Groups & Covering Spaces

2.1 Homotopy

In the sequel $I = [0, 1]$ will denote the unit interval.

Definition 2.1. Let X and Y be topological spaces with continuous maps $f, g\colon X \longrightarrow Y$. A *homotopy* H from f to g is a continuous map $H\colon X \times I \longrightarrow Y$ such that $H(\_, 0) = f$ and $H(\_, 1) = g$. The maps f and g are called *homotopic* if there exists a homotopy from f to g. The maps f and g are called *homotopic relative* A, where A is a subspace of X, if there exists a homotopy H from f to g that fixes A, i. e. such that $H(a, t) = f(a) = g(a)$ for all $t \in I$ and $a \in A$. We will write $f \simeq g$ if f and g are homotopic and $f \simeq_A g$ if f and g are homotopic relative A.

Remark 2.2.
(i) If $f, g\colon X \longrightarrow Y$ are homotopic maps and $h\colon Y \longrightarrow Z$ is continuous, then $h \circ f \simeq h \circ g$.
(ii) Homotopy (relative A) is an equivalence relation. We will write $[f]$ for the *homotopy class (relative A)* of a map $f\colon X \longrightarrow Y$.

Definition 2.3. A map $f\colon X \longrightarrow Y$ is called a *homotopy equivalence* if there exists a map $g\colon Y \longrightarrow X$ such that $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In this case, g is called a *homotopy inverse* to f. The spaces X and Y are called *homotopy equivalent* if there exists a homotopy equivalence $X \longrightarrow Y$; we write $X \simeq Y$.

Remark 2.4. If spaces X and Y are homeomorphic then they are homotopy equivalent.

Write $c_{y_0}\colon X \longrightarrow Y$ for the constant map with value $y_0 \in Y$. If $f \simeq c_{y_0}$ for some $y_0 \in Y$ then we say that f is *null–homotopic*. A space X is called *contractible* if $X \simeq *$, i. e. X is homotopy equivalent to a point. This is the case if and only if id_X is null–homotopic. A null–homotopy of id_X is called a *contraction*.

Examples 2.5. A subset $A \subset \mathbb{R}^n$ is called *star–shaped* with respect to a point $a_0 \in A$ if for all $a \in A$ the line segment $[a_0, a] = \{a_0 + t(a - a_0)\colon t \in [0, 1]\}$ is contained in A. Such subsets are contractible; a contraction is given by $H(a, t) = a_0 + t(a - a_0)$.

Definition 2.6. Two paths $\gamma_0, \gamma_1\colon I \longrightarrow X$ with common endpoints are called *homotopic (as paths)* if $\gamma_0 \simeq_{\partial I} \gamma_1$.

Proposition 2.7.
(i) *If $\gamma\colon I \longrightarrow X$ is a path in X and $\alpha\colon I \longrightarrow I$ is a continuous map fixing the boundary ∂I, then $\gamma \simeq_{\partial I} \gamma \circ \alpha$. The path $\gamma \circ \alpha$ is called a* reparametrisation *of γ.*
(ii) *If $\gamma_1, \gamma_2, \gamma_3\colon I \longrightarrow X$ are paths in X with $\gamma_1(1) = \gamma_2(0)$ and $\gamma_2(1) = \gamma_3(0)$, then concatenation is homotopy associative, i. e. $\gamma_1 * (\gamma_2 * \gamma_3) \simeq_{\partial I} (\gamma_1 * \gamma_2) * \gamma_3$.*
(iii) *Let $\gamma_1, \gamma_2\colon I \longrightarrow X$ and $\gamma_1', \gamma_2'\colon I \longrightarrow X$ be paths in X such that γ_i and γ_i' have the same endpoints for $i = 1, 2$ (separately) and $\gamma_1(1) = \gamma_2(0)$ and such that $\gamma_1' \simeq_{\partial I} \gamma_1$ and $\gamma_2' \simeq_{\partial I} \gamma_2$. Then $\gamma_1 * \gamma_2 \simeq_{\partial I} \gamma_1' * \gamma_2'$.*
(iv) *One has $c_{\gamma(0)} * \gamma \simeq_{\partial I} \gamma \simeq_{\partial I} \gamma * c_{\gamma(1)}$ for any path $\gamma\colon I \longrightarrow X$.*
(v) *One has $\gamma * \gamma^- \simeq_{\partial I} c_{\gamma(0)}$.*

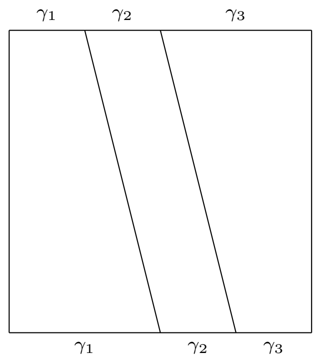


Figure 2: Associativity up to homotopy

Proof.

(i) To show $\gamma \circ \alpha \simeq_{\partial I} \gamma$ it is enough to show that $\alpha \simeq_{\partial I} \mathrm{id}_I$. A homotopy is given by linear interpolation:

$$\begin{aligned} H \colon I \times I &\longrightarrow I \\ (s,t) &\longmapsto t\alpha(s) + (1-t)s. \end{aligned}$$

(ii) One has $\gamma_1 * (\gamma_2 * \gamma_3) = (\gamma_1 * \gamma_2) * \gamma_3 \circ \alpha$ for

$$\alpha(s) = \begin{cases} 1/2 \cdot s, & s \in [0, 1/2] \\ s - 1/4, & s \in [1/2, 3/4] \\ 2s - 1, & s \in [3/4, 1]. \end{cases}$$

(iii) Let H_i be the homotopy from γ_i to γ_i'. Then

$$H(s,t) = \begin{cases} H_1(2s, t), & s \in [0, 1/2] \\ H_2(2s-1, t), & s \in [1/2, 1] \end{cases}$$

yields a homotopy from $\gamma_1 * \gamma_2$ to $\gamma_1' * \gamma_2'$.

(iv) One has $c_{\gamma(0)} * \gamma = \gamma \circ \alpha$ for

$$\alpha(s) = \begin{cases} 0, & s \in [0, 1/2] \\ 2s - 1, & s \in [1/2, 1]. \end{cases}$$

The result follows from (i). An analogous construction works for $\gamma * c_{\gamma(1)}$.

(v) The map $H \colon I \times I \longrightarrow X$ given by

$$H(s,t) = \begin{cases} \gamma(2s(1-t)), & 0 \leq s \leq 1/2 \\ \gamma(2(1-s)(1-t)), & 1/2 \leq s \leq 1 \end{cases}$$

is a homotopy from $\gamma * \gamma^-$ to $c_{\gamma(0)}$. □

Definition 2.8. Let X be a topological space with a fixed base point $x_0 \in X$. Then the set of homotopy classes relative ∂I of closed paths $\gamma\colon I \longrightarrow X$ at x_0, i. e. of *loops* at x_0,

$$\pi_1(X, x_0) = \{[\gamma] : \gamma\colon I \longrightarrow X \text{ a loop at } x_0\}$$

together with the multiplication $[\gamma_1] \cdot [\gamma_2] = [\gamma_1 * \gamma_2]$ is called the *fundamental group* of X at x_0. This product is well–defined by Proposition 2.7.

Remark 2.9. Observe that $\pi_1(X, x_0)$ is in fact a group with neutral element $[c_{x_0}]$.

We will now discuss in which manner $\pi_1(X, x_0)$ depends on the base point $x_0 \in X$. Fix some other point $x_1 \in X$ and let $\tau\colon I \longrightarrow X$ be a path from x_0 to x_1. This path induces a map $c(\tau)\colon \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ which maps a homotopy class $[\gamma] \in \pi_1(X, x_1)$ to $[\tau * \gamma * \tau^-]$. This map is in fact a group homomorphism, because

$$c(\tau)([\gamma_1] \cdot [\gamma_2]) = [\tau * \gamma_1 * \gamma_2 * \tau^-] = [\tau * \gamma_1 * \tau^-] \cdot [\tau * \gamma_2 * \tau^-].$$

The morphism $c(\tau)$ is even isomorphism with inverse $c(\tau^-)$. This shows that the isomorphism class of $\pi_1(X, x_0)$ only depends on the path component of $x_0 \in X$. In particular, if X is path–connected, the isomorphism class of its fundamental group does not depend on the basepoint. The concrete isomorphism $\pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ depends only on the homotopy class of a choice of path from x_0 to x_1. For this reason, we will simply write $\pi_1(X)$ for the fundamental group of a path–connected space if there is no danger of confusion.

Definition 2.10. A path–connected space X is called *simply connected* if $\pi_1(X) = 1$.

Remark 2.11. A space X is simply connected if and only if for any $x_0, x_1 \in X$ there is a unique homotopy class of paths from x_0 to x_1.

The fundamental group is a *functorial* construction in the following sense. Any continuous map $f\colon X \longrightarrow Y$ between topological spaces X and Y induces a map $f_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ such that $f_*([\gamma]) = [f \circ \gamma]$. Because of $f \circ (\gamma_1 * \gamma_2) = (f \circ \gamma_1) * (f \circ \gamma_2)$, this map is a group homomorphism. Given another continuous map $g\colon Y \longrightarrow Z$ to a third space Z, it is immediate that $g_* \circ f_* = (g \circ f)_*$; and of course $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X, x_0)}$.

Additionally, the fundamental group enjoys a kind of *homotopy invariance*. Let $f, g\colon X \longrightarrow Y$ be continuous maps admitting a homotopy $H\colon X \times I \longrightarrow Y$ from f to g. Fix a base point $x_0 \in X$ and write $\tau = H(x_0, \_)$. Then τ is a path from $f(x_0)$ to $g(x_0)$ in Y.

Lemma 2.12. *In this situation, the diagram*

$$\begin{array}{ccc} & \pi_1(X, x_0) & \\ {\scriptstyle f_*} \swarrow & & \searrow {\scriptstyle g_*} \\ \pi_1(Y, f(x_0)) & \xrightarrow[c(\tau)]{\sim} & \pi_1(Y, g(x_0)) \end{array}$$

commutes.

Proof. Take some $[\gamma] \in \pi_1(X, x_0)$. Then $(c(\tau) \circ g_*)([\gamma]) = [\tau * (g \circ \gamma) * \tau^-]$ and $f_*([\gamma]) = [f \circ \gamma]$. We construct a homotopy relative ∂I between these two paths as follows. Write $H_t = H(\_, t)$ and set $K(\_, t) = \tau_t * (H_t \circ \gamma) * \tau_t^-$ where $\tau_t(s) = \tau(ts)$. This defines a homotopy $K\colon I \times I \longrightarrow Y$ relative ∂I from $f \circ \gamma$ to $\tau * (g \circ \gamma) * \tau^-$. □

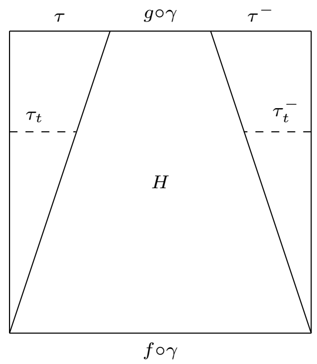


Figure 3: The homotopy K in the proof of Lemma 2.12

Proposition 2.13. *If a continuous map $f\colon X \longrightarrow Y$ is a homotopy equivalence, then the induced map $f_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is an isomorphism.*

Proof. Let $g\colon Y \longrightarrow X$ be a homotopy inverse for f. Lemma 2.12 implies the existence of a commutative diagram

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, f(x_0)) \\ \| & & \big\downarrow g_* \\ \pi_1(X, x_0) & \xrightarrow{\sim} & \pi_1(X, g(f(x_0))), \end{array}$$

i. e. that $g_* \circ f_*$ is a group isomorphism. Hence, f_* is injective. Similarly, there is a commutative square

$$\begin{array}{ccc} \pi_1(Y, f(x_0)) & \xrightarrow{g_*} & \pi_1(X, g(f(x_0))) \\ \| & & \big\downarrow f_* \\ \pi_1(Y, f(x_0)) & \xrightarrow{\sim} & \pi_1(Y, f(g(f(x_0)))) \end{array}$$

which implies that $f_*\colon \pi_1(X, g(f(x_0))) \longrightarrow \pi_1(Y, f(g(f(x_0))))$ is surjective. To fix the apparent issue with base points, observe that there is a commutative diagram

$$\begin{array}{ccc} \pi_1(X, g(f(x_0))) & \xrightarrow{f_*} & \pi_1(Y, f(g(f(x_0)))) \\ \cong \big\uparrow & & \big\uparrow \cong \\ \pi_1(X, x_0) & \xrightarrow[f_*]{} & \pi_1(Y, f(x_0)) \end{array}$$

which implies that the homomorphism $f_*\colon \pi_1(X, x_0) \longrightarrow \pi_1(Y, f(x_0))$ is surjective, too. □

The fundamental group interacts very nicely with products of topological space. Let (X, x_0) and (Y, y_0) be topological space with base points. Then it is easy to check that the map

$$\pi_1(X \times Y, (x_0, y_0)) \longrightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

induced by the projections $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ is an isomorphism.

Examples 2.14.

(i) If X is contractible, then $\pi_1(X) = 1$.
(ii) If $A \subset \mathbb{R}^n$ is star–shaped, then $\pi(A) = 1$.
(iii) $\pi_1(S^n) = 1$ for all $n \geq 2$.

Proof.

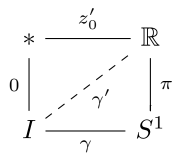
(i) Let X be contractible, so there exists a continuous $H : I \times X \longrightarrow X$ such that $H_0 = \mathrm{id}_X$ and $H_1 = c_{x_0}$. Let $\gamma I \longrightarrow X$ be a loop. Then $\widetilde{H} : I \times I \longrightarrow X$ with $\widetilde{H}(s,t) = H(s, \gamma(t))$ is a homotopy between γ and c_{x_0}.
(ii) Follows from (1).
(iii) Let $\gamma : I \longrightarrow X$ be a loop based at some point $x_0 \in S^n$ and assume there is a $x \in X$ such that $x \notin \gamma(I)$. Then $\gamma : I \longrightarrow X \setminus \{x\} \cong \mathbb{R}^n$ via the stereographic projection. Since $\mathbb{R}^n$ is contractible, we have $\gamma \simeq_{\partial I} c_{x_0}$. What remains to show is that for any loop γ based on x_0, we can find a homotopy to a loop γ' such that $\gamma'(I) \subset X \setminus \{x\}$ for some point $x \in X$. Let $B \cong B^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$ be an open neighbourhood of x such that $x_0 \notin B$. Then $\gamma^{-1}(B) \subset (0,1)$ is a disjoint union of open intervals $(a_i, b_i) \subset (0,1)$. Because $\gamma^{-1}(\{x\})$ is compact there exists a finite subset $J \subset I$ such that $\gamma^{-1}(\{x\})$ is covered by $((a_i, b_i))_{i \in J}$. For every $i \in J$, connect $\gamma(a_i)$ and $\gamma(b_i)$ with a path η_i which is fully contained in ∂B. Because $\overline{B} \cong D^n$ is simply connected, the path η_i will be homotopic to $\gamma|_{[a_i, b_i]}$. Now, let γ' be the loop γ but with the sections on (a_i, b_i) replaced by η_i. Then γ' will not meet x and is homotopic to γ. □

2.2 The Fundamental Group of the Circle

Consider the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. As a matter of convention, we choose $1 \in S^1$ as a base point. We will consider the exponential map $\pi : \mathbb{R} \longrightarrow S^1$ with $\pi(\xi) = e^{2\pi i \xi}$. This map is an example of a *covering map*, a notion that we will defined later. For now, we will prove some characteristic properties of covering maps for π.

Theorem 2.15. *Let $\pi : \mathbb{R} \longrightarrow S^1$ be as above.*

(i) *The map π has the* path lifting property*: Given $z_0 \in S^1$, $z_0' \in \pi^{-1}(z_0) \subset \mathbb{R}$ and any path $\gamma : I \longrightarrow S^1$ with start point z_0 there exists a unique path $\gamma' : I \longrightarrow \mathbb{R}$ starting at z_0' such that $\pi \circ \gamma' = \gamma$, i. e. such that*



commutes. We will usually call γ' a lift *of γ with base point z_0'.*
(ii) *The map π has the* homotopy lifting property*: Given paths $\gamma_0, \gamma_1 : I \longrightarrow S^1$ from z_0 to z_1, a homotopy H from γ_0 to γ_1 and any lift γ_0' of γ_0, there exists a unique homotopy*

$H' \colon I \times I \longrightarrow \mathbb{R}$ such that $H'(\_,0) = \gamma_0'$ and $\pi \circ H' = H$, i. e. such that

$$\begin{array}{ccc} I & \xrightarrow{\gamma_0'} & \mathbb{R} \\ {\scriptstyle \mathrm{id}\times 0}\Big\downarrow & \overset{H'}{\nearrow} & \Big\downarrow{\scriptstyle \pi} \\ I \times I & \xrightarrow[H]{} & S^1 \end{array}$$

commutes. Again, H' will be called a lift *of H.*

Proof.

(i) First, suppose that $\gamma', \gamma'' \colon I \longrightarrow \mathbb{R}$ are two lifts of γ starting at z_0'. Then $\gamma'(t) - \gamma''(t) \in \mathbb{Z}$ for all t and because of continuity, $\gamma' - \gamma''$ will be constant. But $\gamma'(0) = \gamma''(0)$, hence $\gamma' = \gamma''$. To prove existence the existence of a lift γ', observe that for every $\xi \in \mathbb{R}$ the map $\pi_\xi = \pi|_{(\xi,\xi+1)} \colon (\xi, \xi+1) \longrightarrow S^1 \setminus \pi(\xi)$ is homeomorphism. Denote its inverse by s_ξ. Since I is compact, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $\gamma|_{[t_i,t_{i+1}]} \subset S^1 \setminus \{z_i\}$ for all i and some $z_i \in S^1$. Inductively assume $\gamma'|_{[0,t_i]}$ is already constructed. Choose $\xi_i \in \pi^{-1}(z_i)$ such that $\gamma'(t_i) \in (\xi_i, \xi_i + 1)$. Now, set $\gamma'|_{[t_i,t_{i+1}]} = s_{\xi_i} \circ \gamma|_{[t_i,t_{i+1}]}$ and we have defined γ' on $[0, t_{i+1}]$.

(ii) If H' is as required, then $H'(s,\_)$ is the lift of the path H_s starting at $\gamma_0'(s)$. This proves uniqueness. We are left to check is that this defines a continuous map $H \colon I \times I \longrightarrow \mathbb{R}$. Take $s_0 \in I$. Since I is compact, there is a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and a connected neighbourhood U of s_0 such that $H(U \times [t_i, t_{i+1}]) \subset S^1 \setminus \{z_i\}$ for some z_i and all i. Inductively, assume H' is continuous on $U \times [0, t_i]$. Then $H'(U \times \{t_i\}) \subset (\xi_i, \xi_i + 1)$ for some $\xi_i \in \pi^{-1}(z_i)$ and $H' = s_{\xi_i} \circ H$ on $U \times [t_i, t_{i+1}]$, hence H' is continuous on $U \times [0, t_{i+1}]$. ☐

Remark 2.16. Lifts of homotopic paths with the same start point are homotopic and have the same end point. Hence, the map $\varphi \colon \pi_1(S^1) \longrightarrow \mathbb{Z}$ such that $\varphi([\gamma]) = \gamma'(1)$ for some lift γ' of γ starting at 0 is well defined.

Theorem 2.17. *The map $\varphi \colon \pi_1(S^1) \longrightarrow \mathbb{Z}$ of Remark 2.16 is an isomorphism.*

Proof. To prove that φ is a homomorphism, note that a lift of $\gamma_1 * \gamma_2$ is given by $\gamma_1' * \gamma_2'$ where γ_1' is a lift γ_1 starting at 0 and γ_2' is a lift of γ_2 starting at $\gamma_1'(1)$. By continuity, the difference between γ_2' and a lift of γ_2 starting at 0 is constant. This implies $\varphi([\gamma_1][\gamma_2]) = \varphi([\gamma_1]) + \varphi([\gamma_2])$.

The map φ is surjective, because for $n \in \mathbb{Z}$ consider the path $\gamma' \colon I \longrightarrow \mathbb{R}$ with $\gamma'(t) = nt$. Then $\varphi([\pi \circ \gamma']) = n$. It is also injective: Let $\gamma \colon I \longrightarrow S^1$ be a loop such that $\varphi([\gamma]) = 0$. Choose a lift γ' of γ starting at 0. Then γ' is homotopic to c_0 via some homotopy H' in $\mathbb{R}$ and $H = \pi \circ H'$ will be a homotopy from γ to c_1. ☐

Remark 2.18.

(i) From the description of $\pi_1(S^1)$ one immediately deduces $\pi_1(T^2) \cong \pi_1(S^1 \times S^1) \cong \mathbb{Z}^2$.

(ii) Given spaces X and Y, denote the set of homotopy classes of continuous maps $X \longrightarrow Y$ by $[X, Y]$. The circle S^1 becomes a topological group when given the multiplication induced from $\mathbb{C}$. Hence, any set $[X, S^1]$ inherits a canonical group structure; in particular, $[S^1, S^1]$ is a group.

(iii) Consider the restriction $\pi_0 \colon I \longrightarrow S^1$ of the exponential map $\mathbb{R} \longrightarrow S^1$. Given a continuous map $f \colon S^1 \longrightarrow S^1$ consider the loop $g = f \circ \pi_0 \colon I \longrightarrow S^1$ and take some lift $g' \colon I \longrightarrow \mathbb{R}$ along the exponential map. Then $g'(1) - g'(0) \in \mathbb{Z}$ will be independent of the choice of lift g'

and we define $\deg(f) = g'(1) - g'(0)$. One can show that, if $f_0 \simeq f_1$, then $\deg(f_0) = \deg(f_1)$ and that deg descends to an isomorphism $[S^1, S^1] \longrightarrow \mathbb{Z}$.

2.3 The Theorem of Seifert and van Kampen

We will investigate how to calculate the fundamental group of a topological space X admitting an open cover $X = U \cup V$ where U, V and $U \cap V$ are path–connected and nonempty. It turns out that $\pi_1(X, x_0)$, with $x_0 \in U \cap V$, is completely determined by $\pi_1(U, x_0)$, $\pi_1(V, x_0)$ and $\pi_1(U \cap V, x_0)$. To give the explicit connection, we will first need to introduce some constructions on groups.

For groups G_1 and G_2 we define their *free product* $G_1 * G_2$ as the set of reduced words $\omega = \omega_1 \cdots \omega_n$ such that any ω_i is an element of $G_1 \smallsetminus \{e_1\}$ or $G_2 \smallsetminus \{e_2\}$ and neighbouring letters come from different groups. Note that the empty word is allowed. Define an operation on $G_1 * G_2$ by

$$\omega\omega' = \begin{cases} \omega_1 \cdots \omega_n \omega'_1 \cdots \omega'_n & \text{if } \omega_n \text{ and } \omega'_1 \text{ are in different groups,} \\ \omega_1 \cdots \omega_{n-1}(\omega_n \cdot \omega'_1)\omega'_2 \cdots \omega'_n & \text{otherwise.} \end{cases}$$

It is straightforward to check that this defines a group structure on $G_1 * G_2$ with neutral element the empty word. There are obvious maps $\iota_1 \colon G_1 \longrightarrow G_1 * G_2$ and $\iota_2 \colon G_2 \longrightarrow G_1 * G_2$ which induce an isomorphism

$$(\iota_1^*, \iota_2^*) \colon \operatorname{Hom}(G_1 * G_2, H) \xrightarrow{\sim} \operatorname{Hom}(G_1, H) \times \operatorname{Hom}(G_2, H)$$

for any group H. More explicitly, for homomorphisms $\varphi_1 \colon G_1 \longrightarrow H$ and $\varphi_2 \colon G_2 \longrightarrow H$ there exists precisely one homomorphism $\varphi \colon G_1 * G_2 \longrightarrow H$ such that $\varphi \circ \iota_i = \varphi_i$. As usual, this universal property determines $G_1 * G_2$ up to unique isomorphism.

More generally, let G_1, G_2 and A be groups together with homomorphisms $i_i \colon A \longrightarrow G_i$. Denote by N the normal subgroup of $G_1 * G_2$ generated by the elements $i_1(a)i_2(a)^{-1}$ for $a \in A$. We define the *amalgamated free product* $G_1 *_A G_2$ as $(G_1 * G_2)/N$. Again, there are obvious maps $\iota_1 \colon G_1 \longrightarrow G_1 *_A G_2$ and $\iota_2 \colon G_2 \longrightarrow G_1 *_A G_2$ which make

$$\begin{array}{ccc} A & \xrightarrow{i_1} & G_1 \\ {\scriptstyle i_2}\big\downarrow & & \big\downarrow{\scriptstyle \iota_1} \\ G_2 & \xrightarrow[\iota_2]{} & G_1 *_A G_2 \end{array}$$

commute and these maps induce an isomorphism

$$(\iota_1^*, \iota_2^*) \colon \operatorname{Hom}(G_1 *_A G_2, H) \xrightarrow{\sim} \operatorname{Hom}(G_1, H) \times_{\operatorname{Hom}(A,H)} \operatorname{Hom}(G_2, H)$$

for any group H. In other words, for any commutative square

$$\begin{array}{ccc} A & \xrightarrow{i_1} & G_1 \\ {\scriptstyle i_2}\big\downarrow & & \big\downarrow{\scriptstyle \varphi_1} \\ G_2 & \xrightarrow[\varphi_2]{} & H \end{array}$$

there exists a unique homomorphism $\varphi \colon G_1 *_A G_2 \longrightarrow H$ such that $\varphi \circ \iota_i = \varphi_i$. We are now ready to state the Seifert–van Kampen theorem.

Theorem 2.19 (Seifert–van Kampen). *Let X be a topological space admitting an open cover $X = U \cup V$ such that U, V and $U \cap V$ are path–connected and nonempty. Then the inclusions $U \hookrightarrow X$ and $V \hookrightarrow X$ induce an isomorphism*

$$\varphi\colon \pi_1(U, x_0) *_{\pi_1(U\cap V, x_0)} \pi_1(V, x_0) \xrightarrow{\ \sim\ } \pi_1(X, x_0)$$

for any point $x_0 \in U \cap V$.

Proof. We will suppress the base point x_0 for ease of notation. The relevant homomorphism $\varphi\colon \pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V) \longrightarrow \pi_1(X)$ is given by the universal property of $\pi_1(U) *_{\pi_1(U\cap V)} \pi_1(V)$.

To show that φ is surjective consider any loop γ in X based at x_0. We will show that γ is homotopic to a product $\gamma_1 * \cdots * \gamma_n$ of loops lying fully in U or fully in V. Since I is compact, there is a partition $0 = t_0 < \cdots < t_n = 1$ such that $\gamma|_{[t_i, t_{i+1}]}$ is fully in U or fully in V, but not the same for neighbouring intervals. For every $\gamma(t_i)$ choose a path τ_i from $\gamma(t_i)$ to x_0 lying fully in $U \cap V$—this is possible because $U \cap V$ was assumed to be path–connected. Then $\gamma \simeq \gamma_1' * \tau_1 * \tau_1^{-1} * \cdots * \tau_n \tau_n^{-1} * \gamma_n'$ where γ_i' is the path $\gamma|_{[t_{i-1}, t_i]}$ reparametrised to be a path $\gamma_i'\colon I \longrightarrow X$. This is a representation of γ as a product of loops lying fully in U or fully in V.

To see that φ is injective we show that if $\omega = [\gamma_1]\cdots[\gamma_n]$ is a word in $\pi_1(U) * \pi_1(V)$ such that $\gamma_1 * \cdots * \gamma_n \simeq c_{x_0}$, then ω is of the form $\omega_1 i_1(a_1) i_2(a_1)^{-1}\omega_1^{-1} \cdots \omega_k i_1(a_k) i_2(a_k)^{-1}\omega_k^{-1}$ with $a_i \in \pi_1(U \cap V)$. Let $H\colon I \times I \longrightarrow X$ be a homotopy from $\gamma_1 * \cdots * \gamma_n$ to c_{x_0}. Because $I \times I$ is compact there exist finite partitions $0 = t_0 < \cdots < t_n = 1$ and $0 = s_0 < \cdots < s_\ell = 1$ such that $H([s_i, s_{i+1}] \times [t_j, t_{j+1}])$ lies fully in U or V. Without loss of generality we may assume that the γ_i are reparametrisations of $H_0|_{[s_k, s_{k'}]}$ for some $k < k'$. Choose paths τ_{ij} from $H(s_i, t_j)$ to x_0 lying fully in U or V respectively. Let f_{ij} be a reparametrisation of $H(\_, t_j)|_{[s_{i-1}, s_i]}$ and g_{ij} a reparametrisation of $H(s_i, \_)|_{[t_{j-1}, t_j]}$ and write $f_{ij}' = \tau_{i-1,j}^- * f_{ij} * \tau_{ij}$ and $g_{ij}' = \tau_{i,j-1}^- * g_{ij} * \tau_{ij}$. We show that

$$[f_{1j}'] \cdots [f_{n,j}'] \equiv [f_{1,j+1}'] \cdots [f_{n,j+1}'] \pmod N.$$

Then we will find inductively that $\omega = e$. But

$$\begin{aligned}[f_{1j}'] \cdots [f_{ij}'] \cdots [f_{n,j+1}'] &= [f_{1,j+1}' * (g_{2j}')^-] \cdots [g_{i+1,j}' * f_{i,j+1}' * (g_{i,j+1}')^-] \cdots = \\ &= [f_{1,j+1}'][g_{2j}']^{-1}[g_{2j}'] \cdots [g_{nj}']^{-1}[g_{nj}'][f_{n,j+1}']\end{aligned}$$

and a factor $[g_{ij}']^{-1}[g_{ij}']$ is either trivial or an element of N. This proves injectivity of φ. ☐

Definition 2.20. Let X_i, $i \in I$, be based topological spaces, with base points $x_i \in X_i$. The *wedge sum* or *one–point union* of the X_i is

$$\bigvee_{i\in I} X_i = \coprod_{i\in I} X_i \Big/ \coprod_{i\in I} \{x_i\}.$$

Examples 2.21. Let $X = S^1 \vee \cdots \vee S^1$ be a bouquet of n circles. Then $\pi_1(X) = F_n$, the free group on n generators $F_n = \mathbb{Z}^{*n}$.

If G is a finitely generated group, e. g. $G = \langle s_1, \ldots, s_n\rangle$, then $s_1, \ldots, s_n$ define a surjective group homomorphism $\varphi\colon F_n \twoheadrightarrow G$. Hence, $G \cong F_n/N$ for the normal subgroup $N = \ker\varphi \subset F_n$. If N is generated (as a normal subgroup) by elements r_i for $i \in I$, then we write

$$G = \langle s_1, \ldots, s_n \mid (r_i)_{i\in I}\rangle.$$

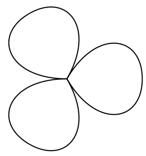


Figure 4: A bouquet of 3 circles

This is called a *presentation* of G. If N is finitely generated as a normal subgroup, then G is called *finitely presented*. For example, if Σ_g is the orientable surface of genus g, it is possible to use the Seifert–van Kampen theorem to compute the fundamental group $\pi_1(\Sigma_g)$. One decompose the $4g$–gon from which Σ_g is obtained into a disk U around the origin and the complement V of a disk strictly contained in U. Then U is contractible and V is homotopy equivalent to $\partial P_{4g}/{\sim}$, the boundary ∂P_{4g} with the identifications made in P_{4g} to obtain Σ_g. Hence, V is homotopy equivalent to a bouquet of $2g$ circles. Seifert–van Kampen implies that $\pi_1(\Sigma_g) \cong 1 *_{\mathbb{Z}} F_{2g}$ where the relevant map $\varphi\colon \pi_1(U \cap V) \longrightarrow \pi_1(V)$ is given by the commutative diagram

$$\begin{array}{ccc} \pi_1(U \cap V) & \xrightarrow{\ \varphi\ } & \pi_1(V) \\ {\scriptstyle\cong}\big| & & \big|{\scriptstyle\cong} \\ \mathbb{Z} & \xrightarrow[\ \psi\ ]{} & F_{2g} \end{array}$$

and $\psi(1) = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$. This implies that we have the presentation

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle.$$

2.4 Covering Spaces

Definition 2.22. A continuous map $\pi\colon E \longrightarrow B$ is called *locally trivial with typical fiber F*, if for all $b \in B$ there exists an open neighbourhood U of b and a homeomorphism $\varphi\colon U \times F \longrightarrow \pi^{-1}(U)$ such that

$$\begin{array}{ccccc} U \times F & \xrightarrow[\varphi]{\sim} & \pi^{-1}(U) & \longrightarrow & E \\ & \searrow{\scriptstyle \mathrm{proj}_U} & \big| & & \big|{\scriptstyle\pi} \\ & & U & \longrightarrow & B \end{array}$$

commutes. The homeomorphism $\varphi^{-1}\colon \pi^{-1}(U) \longrightarrow U \times F$ is called a *local trivialisation* of $\pi\colon E \longrightarrow B$. For any $b \in B$, the subset $\pi^{-1}(\{b\}) \subset E$ is called the *fibre* over b. Observe that every fibre $\pi^{-1}(\{b\})$ is homeomorphic to F.

A locally trivially map $\pi\colon E \longrightarrow B$ with discrete typical fibre F is called a *covering space*. In this case E is called the *total space* and B is called the *base*.

Remark 2.23.

(i) If F is discrete, then $U \times F \cong \coprod_{f \in F} U \times \{f\}$.

(ii) The projection $\mathrm{proj}_U\colon U \times F \longrightarrow U$ restricts to homeomorphisms $U \times \{f\} \longrightarrow U$. Hence, the covering space $\pi\colon E \longrightarrow B$ is a local homeomorphism.

(iii) If $|F| = n$ is finite, then a covering space $\pi\colon E \longrightarrow B$ with typical fibre F is called *n–sheeted*.

Examples 2.24.

(i) We have already encountered a covering space, namely the exponential map $\mathbb{R} \longrightarrow S^1$. Its typical fibre is $\mathbb{Z}$.

(ii) For every $n \geq 1$ there is an n–sheeted covering space $\pi\colon S^1 \longrightarrow S^1$ such that $\pi(z) = z^n$. Here, S^1 is considered to be a subset of $\mathbb{C}$.

Further examples of covering spaces may be obtained by certain "good" group actions.

Definition 2.25. A group action $G \times X \longrightarrow X$ of a group G on a space X is called *covering space action* or *properly discontinuous*, if for all points $x \in X$ there exists an open neighbourhood U of x such that $gU \cap U = \emptyset$ for all $g \in G \setminus \{e\}$.

Remark 2.26.

(i) Any covering space action is *free*, i. e. for all $x \in X$ and $g \neq e \in G$ one has $gx \neq x$.

(ii) The converse to (i) is false. For $z_0 = \exp(2\pi i \xi_0) \in S^1$ with $\xi_0 \in \mathbb{R} \setminus \mathbb{Q}$ the action of $\mathbb{Z}$ on S^1 given by $n.z = z_0^n z$ is free but not a covering space action. In fact, every orbit is dense.

(iii) Any free action of a *finite* group on a *Hausdorff* space is a covering space action.

Proposition 2.27. *If $G \times X \longrightarrow X$ is a covering space action, then the canonical projection $\pi\colon X \longrightarrow X/G$ is a covering space with typical fibre G.*

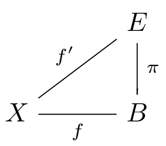
Proof. The projection $\pi\colon X \longrightarrow X/G$ is an open map, since for an open subset $U \subset X$ the image $\pi(U)$ satisfies $\pi^{-1}(\pi(U)) = \bigcup_{g\in G} gU$ and $gU \subset X$ is open for any $g \in G$. Take any $x \in X$ and choose some open neighbourhood U of x such that $gU \cap U = \emptyset$ for all $g \neq e$. For such a U the restriction $\pi|_U\colon U \longrightarrow V = \pi(U)$ is a homeomorphism because it is continuous, open and bijective. Denote the inverse by $s\colon V \longrightarrow U$. Then a local trivialisation over U is given by the map $\varphi\colon V \times G \longrightarrow \pi^{-1}(V)$ with $\varphi(y, g) = g \cdot s(y)$. □

Examples 2.28.

(i) The group $\mathbb{Z}^\times$ acts freely on S^n. Hence, $\pi\colon S^n \longrightarrow S^n/\mathbb{Z}^\times \cong \mathbb{RP}^n$ is a 2–sheeted covering space. We will see soon, that this implies that $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}^\times \cong \mathbb{Z}/2$.

(ii) The additive group $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by translation. This is a covering space action and $\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$.

Definition 2.29. Let $\pi\colon E \longrightarrow B$ be a covering space and consider any continuous map $f\colon X \longrightarrow B$. A continuous map $f'\colon X \longrightarrow E$ such that



commutes is called a *lift* of f to E.

Proposition 2.30. *Let $\pi\colon E \longrightarrow B$ be a covering space and $f\colon X \longrightarrow B$ a continuous map. Let $x_0 \in X$. and consider lifts $f', f''\colon X \longrightarrow E$ of f with $f'(x_0) = f''(x_0)$. If X is connected, then $f' = f''$.*

Proof. We show that the set $A = \{x \in X : f'(x) = f''(x)\} \subset X$ is open and closed. To see that A is open, consider any point $x \in A$ and choose a local trivialisation $\varphi\colon \pi^{-1}(V) \longrightarrow V \times F$ over some neighbourhood V of $f(x)$. Then $\varphi(f'(x)) = \varphi(f''(x)) \in V \times \{a\}$ for some $a \in F$. Because f' and f'' are continuous, there exists some open neighbourhood U of x such that $\varphi(f'(U)) \subset V \times \{a\}$ and $\varphi(f''(U)) \subset V \times \{a\}$. Because π restricts to a homeomorphism $\varphi^{-1}(V \times \{a\}) \longrightarrow V$, this implies that $f'|_U = f''|_U$.

To see that A is closed, take some $x \in X \setminus A$. Let $\varphi\colon \pi^{-1}(V) \longrightarrow V \times F$ be as above. Then $\varphi(f'(x)) \in V \times \{a\}$ and $\varphi(f''(x)) \in V \times \{b\}$ for $a \neq b$. Because f' and f'' are continuous, there exists an open neighbourhood U of x such that $\varphi(f'(U)) \subset V \times \{a\}$ and $\varphi(f''(U)) \subset V \times \{b\}$. This implies that $U \subset X \setminus A$. □

Proposition 2.31. *A covering space $\pi\colon E \longrightarrow B$ satisfies the* homotopy lifting property*, i. e. for any homotopy $H\colon X \times I \longrightarrow B$ from f to g and any chosen lift $f'\colon X \longrightarrow E$ of f there exists a unique lift $H'\colon X \times I \longrightarrow E$ such that*

$$\begin{array}{ccc} X & \xrightarrow{f'} & E \\ {\scriptstyle \mathrm{id}\times 0}\downarrow & \overset{H'}{\nearrow} & \downarrow{\scriptstyle \pi} \\ X \times I & \xrightarrow[H]{} & B \end{array}$$

commutes. In particular $H' \circ (\mathrm{id} \times 1) =: g'$ is a lift of g.

Proof. We will call an open subset $U \subset B$ *admissible* if there exists a local trivialisation over U. Fix $x \in X$. Then there exists an open neighbourhood V_x of x and a subdivision $0 = t_0 < \cdots < t_n = 1$ such that $H(V_x \times [t_{i-1}, t_i]) \subset U_i$ for some admissible U_i. Inductively assume that H' is already constructed on $V_x \times [0, t_i]$. By assumption $H(V_x \times [t_i, t_{i+1}]) \subset U$ for some admissible U and in particular $H'(V_x \times \{t_i\}) \subset \pi^{-1}(U)$. For $a \in F$ write $V_{x,a} = H'(\_, t_i)^{-1}(U \times \{a\}) \cap V_x$, where we identify $U \times \{a\}$ with $\varphi^{-1}(U \times \{a\})$. Observe that π restricts to a homeomorphism $U \times \{a\} \longrightarrow U$ and define $H'\colon V_{x,a} \times [t_i, t_{i-1}] \longrightarrow U \times \{a\}$ by $H'(y,t) = \pi^{-1}(H(y,t))$. Inductively, one obtains a lift $H'\colon V_x \times I \longrightarrow E$. Proposition 2.30 implies that such lifts are unique and in particular the constructed lifts coincide on $(V_x \times I) \cap (V_{x'} \times I)$ for $x, x' \in X$. Hence, we arrive at a lift $H'\colon X \times I \longrightarrow E$. □

As special cases of the previous proposition we obtain:

Proposition 2.32. *Let $\pi\colon E \longrightarrow B$ be a covering map.*

(i) *For $b_0 \in B$, $e_0 \in \pi^{-1}(\{b_0\})$ and a path $\gamma\colon I \longrightarrow B$ with $\gamma(0) = b_0$ there is a unique lift γ' in*

$$\begin{array}{ccc} * & \xrightarrow{e_0} & E \\ {\scriptstyle 0}\downarrow & \overset{\gamma'}{\nearrow} & \downarrow{\scriptstyle \pi} \\ I & \xrightarrow[\gamma]{} & B. \end{array}$$

(ii) *Consider paths $\gamma_0, \gamma_1\colon I \longrightarrow B$ and a homotopy H relative ∂I from γ_0 to γ_1. For any lift $\gamma_0'\colon I \longrightarrow E$ of γ_0 there exists a unique lift $H'\colon I \times I \longrightarrow E$ of H to E which yields a homotopy relative ∂I from γ_0' to a lift of γ_1.*

2.5 Fundamental Groups and Covering Spaces

Let $\pi \colon E \longrightarrow B$ be a covering map of pointed spaces, i. e. a covering space together with fixed base points $b_0 \in B$ and $e_0 \in \pi^{-1}(\{b_0\})$.

Proposition 2.33. *The homomorphism $\pi_* \colon \pi_1(E, e_0) \longrightarrow \pi_1(B, b_0)$ induced by π is injective.*

Proof. Take a loop γ in B such that $\pi_*([\gamma]) = e \in \pi_1(B, b_0)$, i. e. $\pi \circ \gamma \simeq_{\partial I} c_{b_0}$ via a homotopy H. Lifting this homotopy to B gives a homotopy $\gamma \simeq_{\partial I} c_{e_0}$. □

Definition 2.34. The subgroup $\pi_*(\pi_1(E, e_0)) \subset \pi_1(B, b_0)$ is called the *characteristic subgroup* of $\pi \colon E \longrightarrow B$.

Consider the *lifting problem* for a map $f \colon X \longrightarrow B$, that is the problem of finding a map $f' \colon X \longrightarrow E$ such that $\pi \circ f = f'$. We have already proven that such a lift is unique up to a choice of basepoint if X is connected. Moreover, the existence of a lift only depends on the homotopy class of f: if $f \simeq g$ and there exists a lift of f then there also exists a lift of g.

We have a necessary condition for the existence of a lift f'. If

$$\begin{array}{ccc} & & (E, e_0) \\ & \overset{f'}{\nearrow} & \downarrow \pi \\ (X, x_0) & \xrightarrow[f]{} & (B, b_0) \end{array}$$

is a commutative diagram of pointed spaces, then passing to fundamental groups we get a commutative diagram

$$\begin{array}{ccc} & & \pi_1(E, e_0) \\ & \overset{f'_*}{\nearrow} & \downarrow \pi_* \\ \pi_1(X, x_0) & \xrightarrow[f_*]{} & \pi_1(B, b_0). \end{array}$$

Hence, the image of f_* must lie in the characteristic subgroup of $\pi \colon E \longrightarrow B$. Perhaps surprisingly, this condition is also sufficient for nice enough spaces.

Proposition 2.35. *If X is path–connected and locally path–connected, then there exists a lift $f' \colon X \longrightarrow E$ of f if and only if* $\operatorname{im} f_* \subset \operatorname{im} \pi_*$.

Proof. For every point $x \in X$ choose a path $\tau_x \colon I \longrightarrow X$ from x_0 to x and consider the unique lift $\gamma_x \colon I \longrightarrow E$ of $f \circ \tau_x$ to E starting at e_0. Now set $f'(x) = \gamma_x(1)$. In fact, this definition is independent of the choice of τ_x: For any other path τ'_x from x_0 to x, consider the loop $\tau'_x * \tau_x^-$ at x_0. Then on fundamental groups $[(f \circ \tau'_x) * (f \circ \tau_x^-)] = f_*([\tau'_x * \tau_x^-]) \in \pi_*(\pi_1(E, e_0))$. Hence, $(f \circ \tau'_x) * (f \circ \tau'_x)$ lifts to a loop in E at e_0; this implies that $f'(x)$ was in fact independent of the choice of τ_x.

The only thing left to check is that f' is continuous. Let $V \subset B$ be an open neighbourhood of $f(x)$ such that $\pi \colon E \longrightarrow B$ is trivial over V and let $U \subset E$ be an open neighbourhood of $f'(x)$ such that $\pi(U) = V$. Fix any open, path–connected neighbourhood $W \subset X$ of x such that $f(W) \subset V$. For $y \in W$ choose a path η_y from x to y. Then $\tau_x * \eta_y$ is a path from x_0 to y, hence $f'(y) = \eta'(1)$ for a lift η' of $f \circ (\tau_x \circ \eta_y)$ to E starting at $f'(x)$. On the other hand $\gamma_x * (\pi|_U^{-1} \circ f \circ \eta_y)$ is a lift of $f \circ (\tau_x * \eta_y)$. This implies that $f'(y) \in U$ which proves continuity of f'. □

Remark 2.36. One cannot get rid of the assumption that X be locally path–connected.

There is a general notion of *fibre transport*. Given a covering space $\pi\colon E \longrightarrow B$, fix points $b_0, b_1 \in B$. Let B be path–connected and choose a path $\gamma\colon I \longrightarrow B$ from b_0 to b_1. Define a map $T_\gamma\colon \pi^{-1}(\{b_0\}) \longrightarrow \pi^{-1}(\{b_1\})$ via $T_\gamma(e) = \gamma'_e(1)$ where γ'_e is the lift of γ to E starting at e. This map only depends on the homotopy class of γ relative ∂I. This construction satisfies $T_{c_b} = \mathrm{id}_{\pi^{-1}(\{b\})}$ for any $b \in B$ and $T_{\gamma_1 * \gamma_2} = T_{\gamma_2} \circ T_{\gamma_1}$. This latter property immediately implies that T_γ is a bijection for any path γ in B. If $b_1 = b_0$ we obtain a right action of $\pi_1(B, b_0)$ on $\pi^{-1}(\{b_0\})$.

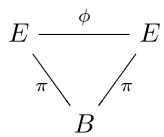
Remark 2.37.

(i) The stabiliser of $e_0 \in \pi^{-1}(\{b_0\})$ under this action is just the characteristic subgroup $\operatorname{im} \pi_*$.
(ii) The action is transitive if E is path–connected.

2.6 Deck Transformations

Let $\pi\colon E \longrightarrow\!\!\!\!\rightarrow B$ be a covering space. We will investigate what the "symmetries" of such an object are.

Definition 2.38. A homeomorphism $\phi\colon E \longrightarrow E$ such that



commutes is called a *deck transformation*. In other words, a deck transformation is a fibre–preserving homeomorphism.

The deck transformation forms a group. We will denote it by $\mathrm{Deck}(\pi)$ or $\mathrm{Deck}_B(E)$.

Remark 2.39. If E is connected and $\phi, \phi'\colon E \longrightarrow E$ are deck transformations which satisfy $\phi(e) = \phi'(e)$ for some point $e \in E$, then $\phi = \phi'$.

Proof. This follows from the uniqueness of lifts because ϕ and ϕ' are both lifts of $\pi\colon E \longrightarrow B$ to E. ☐

Examples 2.40.

(i) Consider the exponential map $\pi\colon \mathbb{R} \longrightarrow\!\!\!\!\rightarrow S^1$. All deck transformations are of the form $\phi_n(\xi) = \xi + n$ with $n \in \mathbb{Z}$, since given $k, l \in \mathbb{Z} = \pi^{-1}(\{1\})$ there exists a ϕ_n such that $\phi_n(k) = l$. The uniqueness statement of the previous remark allows us to conclude. We obtain an isomorphism $\mathbb{Z} \xrightarrow{\sim} \mathrm{Deck}(\pi)$.
(ii) For $n \in \mathbb{N}$ consider the covering n–sheeted covering $\pi\colon S^1 \longrightarrow S^1$ such that $\pi(z) = z^n$. All deck transformations are given by $\phi_k(z) = e^{2\pi i k/n}$ with $k \in \mathbb{Z}/n$. We obtain an isomorphism $\mathbb{Z}/n \xrightarrow{\sim} \mathrm{Deck}(\pi)$.

Definition 2.41. A covering space $\pi\colon E \longrightarrow B$ is called *regular* (or *normal* or *Galois*) if $\mathrm{Deck}(\pi)$ acts transitively on every fibre $\pi^{-1}(\{b_0\})$.

Fix $b_0 \in B$ and $e_0, e_0' \in \pi^{-1}(\{b_0\})$. When is there a deck transformation $\phi\colon E \longrightarrow E$ with $\phi(e_0) = e_0'$? Assume that E is connected and locally path–connected. Then Proposition 2.35 implies that such a deck transformation ϕ exists if and only if $\pi_*(\pi_1(E, e_0)) = \pi_*(\pi_1(E, e_0'))$.

Proposition 2.42. *Let $\pi\colon (E, e_0) \longrightarrow\!\!\!\!\rightarrow (B, b_0)$ be a pointed covering space with E path–connected and locally path–connected. Then there is an isomorphism*

$$\mathrm{N}(\pi_*\pi_1(E, e_0)) \big/ \pi_*\pi_1(E, e_0) \xrightarrow{\sim} \mathrm{Deck}(\pi)$$

where $\mathrm{N}(\pi_*\pi_1(E, e_0))$ denotes the normaliser of $\pi_*\pi_1(E, e_0)$ in $\pi_1(B, b_0)$

Remark 2.43. We have already seen that under the assumptions of Proposition 2.42 the covering space $\pi\colon E \longrightarrow\!\!\!\!\rightarrow B$ is regular if and only if $\pi_*\pi_1(E, e_0)$ is a normal subgroup in $\pi_1(B, b_0)$.

Corollary 2.44. *If under the assumptions of Proposition 2.42 the covering space $\pi\colon E \longrightarrow\!\!\!\!\rightarrow B$ is regular, then*

$$\mathrm{Deck}(\pi) \cong \pi_1(B, b_0) \big/ \pi_*\pi_1(E, e_0).$$

Proof of Proposition 2.42. Define a map $\varphi\colon \mathrm{N}(\pi_*\pi_1(E, e_0)) \longrightarrow \mathrm{Deck}(\pi)$ via $\varphi([\gamma]) = \phi$ where ϕ denotes the unique deck transformation with $\phi(e_0) = \gamma'(1) = e_0'$ for a lift γ' of γ to E. Because of

$$\varphi([\gamma_1][\gamma_2])(e_0) = \eta'(1) = (\gamma_1' * (\varphi([\gamma_1]) \circ \gamma_2'))(1) = \varphi([\gamma_1])(\gamma_2'(1)) = (\varphi([\gamma_1]) \circ \varphi([\gamma_2]))(e_0)$$

for some lift η' of $\gamma_1 * \gamma_2$ starting at e_0, the map φ is a group homomorphism.

Furthermore, φ is surjective because, for $\psi \in \mathrm{Deck}(\pi)$, choose a path τ from e_0 to $e_0' = \psi(e_0)$. Then $\gamma = \pi \circ \tau$ satisfies $[\gamma] \in \mathrm{N}(\pi_*\pi_1(E, e_0))$ and $\varphi([\gamma]) = \psi$.

It remains to compute $\ker \varphi$. We have $\varphi([\gamma]) = \mathrm{id}$ if and only if $\gamma'(0) = \gamma'(1)$ for some lift γ' of γ, i. e. γ lifts to a loop in E. This is equivalent to $[\gamma] \in \pi_*\pi_1(E, e_0)$. □

Proposition 2.45. *Let $G \times X \longrightarrow X$ be a covering space action. Then*

(i) *the covering space $\pi\colon X \longrightarrow X/G$ is regular.*
(ii) *if X is connected,* $\mathrm{Deck}(\pi) \cong G$.
(iii) *if X is path–connected and locally path–connected, there is an isomorphism*

$$\pi_1(X/G, Gx_0) \big/ \pi_*\pi_1(X, x_0) \xrightarrow{\sim} G$$

for any $x_0 \in X$.

Proof.

(i) Clearly, G may be considered as a subgroup of $\mathrm{Deck}(\pi)$ because $\pi(gx) = \pi(x)$ for all $g \in G$. Because G acts transitively on each orbit, this implies that π is regular.
(ii) The argument of (i) implies that $G = \mathrm{Deck}(\pi)$ if X is connected.
(iii) This is just a formal consequence of (i) and (ii) in light of Proposition 2.42. □

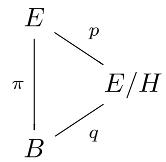
Examples 2.46. Real projective space is given as $\mathbb{RP}^n \cong S^n/\mathbb{Z}^\times$. The action of $\mathbb{Z}^\times$ is a covering space action; hence Proposition 2.45 implies that $\pi_1(\mathbb{RP}^n)/\pi_1(S^n) \cong \mathbb{Z}^\times \cong \mathbb{Z}/2$. So, $\pi_1(\mathbb{RP}^n) \cong \mathbb{Z}/2$ for $n > 1$. On the other hand, $\pi_1(\mathbb{RP}^1) \cong \mathbb{Z}$.

We have seen that a covering space action of G on a connected space X is nothing but the action of $\mathrm{Deck}(\pi)$ on the covering space $\pi\colon X \longrightarrow\!\!\!\!\!\rightarrow X/G$. Conversely, we have the following

Proposition 2.47. *Let $\pi\colon E \twoheadrightarrow B$ be a covering space. Consider any subgroup $H \subset \mathrm{Deck}(\pi)$. Then*

(i) *if E is connected, the action of H on E is a covering space action and $p\colon E \twoheadrightarrow E/H$ is a regular covering space.*

(ii) *the covering space π factors as*



and, if B is connected and locally connected, $q\colon E/H \longrightarrow B$ is a covering space.

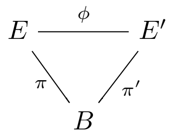
Proof. Take $e \in E$ and $\phi \in \mathrm{Deck}(\pi)$. Let $V \subset B$ be an admissible open neighbourhood of $\pi(e)$ and $U \subset E$ an open neighbourhood of e such that $\pi\colon U \longrightarrow V$ is a homeomorphism. We will prove that $\phi(U) \cap U \neq \emptyset$ implies $\phi = \mathrm{id}$. If $e' \in \phi(U) \cap U$, then $e' = \phi(e'')$ for some $e'' \in U$. Hence, $\pi(e') = \pi(\phi(e'')) = \pi(e'')$ and $e' = e''$ because π is a homeomorphism. The standard uniqueness statement for deck transformations implies $\phi = \mathrm{id}$. This proves (i).

For (ii), let $V \subset B$ an admissible and connected open subset. Write $\pi^{-1}(V) = \coprod_{i \in I} U_i$ such that $\pi|_{U_i}\colon U_i \longrightarrow V$ is a homeomorphism and the U_i are connected components of $\pi^{-1}(V)$. The group H permutes the U_i (since these are the connected components); hence, the H–orbits of the components U_i are open in E/H and map bijectively to V. The map q is continuous and open, so $q\colon E/H \longrightarrow B$ is trivial over V. In summary, q is locally trivial and, because B is connected, the typical fibre is independent of V. □

2.7 Classification of Covering Spaces

Let B be a fixed topological space. We want to classify all possible covering spaces $E \twoheadrightarrow B$ up to isomorphism.

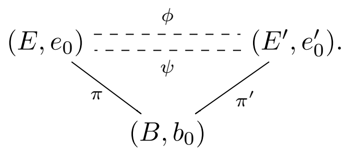
Definition 2.48. Let $\pi\colon E \twoheadrightarrow B$ and $\pi'\colon E' \twoheadrightarrow B$ be covering spaces over B. A homeomorphism $\phi\colon E \longrightarrow E'$ such that



commutes is called an *isomorphism* of covering spaces.

We will see that under suitable assumptions there is a natural bijective correspondence between isomorphism classes of pointed covering spaces $\pi\colon (E, e_0) \twoheadrightarrow (B, b_0)$ and subgroups $H \subset \pi_1(B, b_0)$ which maps a covering to its characteristic subgroup. In fact, if $\mathrm{im}\,\pi_* = \mathrm{im}\,\pi'_*$ for pointed covering spaces $\pi\colon (E, e_0) \twoheadrightarrow (B, b_0)$ and $\pi'\colon (E', e'_0) \twoheadrightarrow (B, b_0)$, then (E, e_0)

and (E', e'_0) are isomorphic, if we assume that E and E' are path–connected and locally path–connected. The isomorphism may be constructed as follows: Consider the diagram



Because $\operatorname{im}\pi_* \subset \operatorname{im}\pi'_*$ there exists a unique map ϕ making this triangle commutative. Analogously, we find a map ψ making the respective triangle commute and the uniqueness of lifts implies that ϕ and ψ are mutually inverse.

To see surjectivity we need to show that for every subgroup $H \subset \pi_1(B, b_0)$ there exists a pointed covering $\pi\colon (E, e_0) \twoheadrightarrow (B, b_0)$ such that $H = \operatorname{im}\pi_*$. In particular, we need to find a covering π such that $\operatorname{im}\pi_* = 1$.

Definition 2.49. Let B be a topological space. A covering space $\pi\colon E \twoheadrightarrow B$ is called *universal* if E is simply connected.

Remark 2.50. If B is locally path–connected, then a universal covering of B is unique up to isomorphism, because in this case any covering of B is locally path–connected and one can apply the argument above.

Definition 2.51. A topological space X is called *semi–locally simply connected* if any point $x \in X$ admits an open neighbourhood U such that any loop $\gamma\colon I \longrightarrow U$ at x is null–homotopic in X.

Remark 2.52. There exists spaces that are path–connected and locally path–connected but not semi–locally simply connected. The *Hawaiian earrings* $\bigcup_{n\in\mathbb{N}} \partial B_{1/n}(0, 1/n) \subset \mathbb{R}^2$ provide an example.

Proposition 2.53. *Every path–connected, locally path–connected and semi–locally simply connected space B admits a universal covering $\pi\colon E \twoheadrightarrow B$.*

Proof. To give some motivation for the construction consider any universal covering $\pi\colon E \longrightarrow B$ with $b_0 \in B$ and $e_0 \in \pi^{-1}(\{b_0\})$. Then for any $e_1 \in E$ there exists a unique homotopy class of paths $[\gamma]$ from e_0 to e_1. Hence, the points of E may be identified with homotopy classes of paths in E starting at e_0. The composition $\pi \circ \gamma$ is a path in B from b_0 to $\pi(e_1) = b_1$. The homotopy lifting property implies that path γ and γ' from e_0 to e_1 are homotopic if and only if the paths $\pi\circ\gamma$ and $\pi\circ\gamma'$ from b_0 to b_1 are homotopic. Hence, points in E can be identified with homotopy classes of paths in B starting at $b_0 \in B$.

To prove the proposition fix $b_0 \in B$. Let E be the set of homotopy classes of paths $\gamma\colon I \longrightarrow B$ with $\gamma(0) = b_0$. For $[\gamma] \in E$ set $\pi([\gamma]) = \gamma(1)$. This defines a surjective map $\pi\colon E \longrightarrow B$ since B is path–connected. The second step is to define a topology on E. For $\gamma\colon I \longrightarrow B$ with $\gamma(0) = b_0$ and an open neighbourhood U of $\gamma(1)$ set

$$U_\gamma = \{[\gamma * \tau]\colon \tau\colon I \longrightarrow U \text{ and } \tau(0) = \gamma(1)\} \subset E.$$

Clearly, U_γ depends only on the homotopy class of γ. We claim that the sets $U_{[\gamma]}$ form a basis of a topology on E: If $[\gamma''] \in U_{[\gamma]} \cap U'_{[\gamma']}$ then $U''_{[\gamma'']} \subset U_{[\gamma]} \cap U'_{[\gamma']}$ for an open neighbourhood $U'' \subset U \cap U'$ of $\gamma''(1)$.

It follows that $\pi\colon E \longrightarrow B$ is continuous an open: Let $U \subset B$ be an open neighbourhood of $\pi([\gamma]) = \gamma(1)$ for some $[\gamma] \in E$. Then $U_{[\gamma]} \subset E$ is open and $\pi([\gamma * \tau]) = \tau(1) \in U$ for any homotopy class $[\gamma * \tau] \in U_{[\gamma]}$. To see that π is open, observe that $\pi(U_{[\gamma]})$ is the path–connected component of U containing $\gamma(1)$ which is open because B is locally path–connected. Since the sets $U_{[\gamma]}$ form a basis, this implies that π is an open map.

Let $b \in B$. There exists an open neighbourhood U of b which is path–connected and such that any loop $\gamma\colon I \longrightarrow U$ is null–homotopic in B. Then

$$\pi^{-1}(U) = \coprod_{[\gamma] \text{ s. t. } \gamma(1) \in U} U_{[\gamma]} :$$

We have $U_{[\gamma]} \cap U_{[\gamma']} = \emptyset$ for $[\gamma] \neq [\gamma']$ since if there were some $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$ then $\gamma'' \simeq \gamma * \tau$ and $\gamma'' \simeq \gamma * \tau'$ for some τ and τ'. But because any loop in U is null–homotopic in B this would imply that $\gamma \simeq \gamma'$. Additionally, for $[\gamma * \tau]$ one clearly has $\pi([\gamma * \tau]) = \tau(1) \in U$ and conversely, if $[\gamma] \in \pi^{-1}(U)$, then $\pi([\gamma]) = \gamma(1) \in U$ and because U is path–connected there exists a path $\tau\colon I \longrightarrow U$ from b to $\gamma(1)$. Then $\gamma \simeq (\gamma * \tau^-) * \tau$, i. e. $[\gamma] \in U_{[\gamma * \tau^-]}$.

Now, $\pi|_{U_{[\gamma]}}\colon U_{[\gamma]} \longrightarrow U$ is injective for suppose $\eta(1) = \eta'(1)$ for paths $[\eta], [\eta'] \in U_{[\gamma]}$, e. g. $\eta \simeq \gamma * \tau$ and $\eta' \simeq \gamma * \tau'$. Then $\eta \simeq \gamma * \tau' * \tau'^- * \tau \simeq \eta'$ because any loop in U is null–homotopic in B. In summary, we have constructed a local trivialisation of $\pi\colon E \longrightarrow B$ around b with discrete fibre.

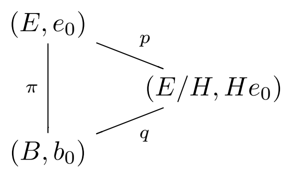
The space E is path–connected: Write $e_0 = [c_{b_0}]$. Take any $[\gamma] \in E$ and for $s \in I$ let $\gamma_s\colon I \longrightarrow B$ be the path such that $\gamma_s(t) = \gamma(st)$. Then the map $\alpha\colon I \longrightarrow E$ such that $\alpha(s) = [\gamma_s]$ is a path from e_0 to $[\gamma]$ in E.

To see that E is simply connected let $\gamma'\colon I \longrightarrow E$ be a loop based at e_0. Write $\gamma = \pi \circ \gamma'$. As above consider $\alpha\colon I \longrightarrow E$ with $\alpha(s) = [\gamma_s]$. Then $\pi(\alpha(s)) = \gamma(s)$ and $\alpha(0) = e_0$. This implies $\alpha = \gamma'$ because π is a covering. In particular, α is a loop in e_0, i. e. $[\gamma_0] = [\gamma_s]$. Hence, γ is null–homotopic. The homotopy lifting property allows us to conclude. ☐

Examples 2.54.

(i) The exponential map $\mathbb{R} \twoheadrightarrow S^1$ is a universal covering of S^1.
(ii) Similarly, the exponential map $\mathbb{C} \twoheadrightarrow \mathbb{C}^\times$ is a universal covering of $\mathbb{C}$.
(iii) The quotient map $\mathbb{R}^2 \twoheadrightarrow T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is a universal covering of the torus.
(iv) The quotient map $S^n \twoheadrightarrow \mathbb{RP}^n = S^n/\mathbb{Z}^\times$ is a universal covering of real projective space.

Let (B, b_0) be a path–connected and locally path–connected pointed space, hence admitting a universal covering $\pi\colon (E, e_0) \twoheadrightarrow (B, b_0)$. To classify all coverings of B, it remains to show that for any subgroup $H \subset \pi_1(B, b_0)$ there exists a covering $\pi'\colon (E', e_0') \twoheadrightarrow (B, b_0)$ with characteristic subgroup. But Proposition 2.47 implies that the action of H on E by deck transformations is a covering space action. We obtain a commutative triangle



of coverings with p regular. We claim that $q_*\pi_1(E/H, He_0) = H$: By definition $q_*\pi_1(E/H, He_0)$ are the homotopy classes of loops γ based at b_0 which lift to loops based at He_0 in the covering space E/H. But this is equivalent to γ lifting to a loop in E which is based in the H–orbit in e_0, i. e. $[\gamma] \in H$.

Furthermore, the constructed covering $\pi'\colon (E', e'_0) \longrightarrow (B, b_0)$ with characteristic subgroup H is regular if and only if H is a normal subgroup of $\pi_1(B, b_0)$. In summary, we have proven the following proposition.

Proposition 2.55. *If B is path–connected, locally path–connected and semi–locally simply connected, then there is a natural bijective correspondence*

$$\left\{ \begin{array}{c} \textit{Pointed covering spaces} \\ \pi\colon (E, e_0) \twoheadrightarrow (B, b_0) \textit{ up to isomorphism} \end{array} \right\} \text{---} \{\textit{Subgroups } H \subset \pi_1(B, b_0)\}.$$

Furthermore, $\pi\colon (E, e_0) \twoheadrightarrow (B, b_0)$ is regular if and only if the corresponding subgroup H is a normal subgroup of $\pi_1(B, b_0)$.

Corollary 2.56. *Under the same conditions on B, there is a natural bijective correspondence*

$$\left\{ \begin{array}{c} \textit{Covering spaces } \pi\colon E \twoheadrightarrow B \textit{ up to} \\ \textit{isomorphism} \end{array} \right\} \text{---} \left\{ \begin{array}{c} \textit{Subgroups } H \subset \pi_1(B, b_0) \textit{ up to} \\ \textit{conjugacy} \end{array} \right\}.$$

In particular, regular covering spaces correspond to normal subgroups in $\pi_1(B, b_0)$.

Remark 2.57. There is a formal analogy with Galois theory. If $L|k$ is a Galois extension, there is a bijective correspondence between intermediate extensions $K|k$ and subgroups $H \subset \mathrm{Gal}(L|k)$. Here, normal subgroups correspond to Galois extensions. Instead of the quotient by a group action one considers the fixed field of H in L.

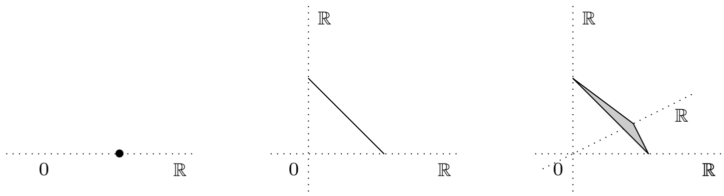


Figure 5: The standard 0-simplex, 1-simplex and 2-simplex

3 Singular Homology Theory

3.1 Singular Simplices and the Singular Chain Complex

Definition 3.1. The *standard n–simplex* $\Delta^n \subset \mathbb{R}^{n+1}$ is the convex hull of the standard basis $e_0, \ldots, e_n$, i. e.

$$\Delta^n = \left\{ \sum_{i=0}^{n} t_i e_i \colon 0 \leq t_i \leq 1 \text{ and } \sum_{i=0}^{n} t_i = 1 \right\}.$$

In this presentation the t_i are called *barycentric coordinates.*

Remark 3.2. One can show that $\Delta^n \cong D^n$.

Definition 3.3. Let X be a topological space. A continuous map $\sigma \colon \Delta^n \longrightarrow X$ is called a *singular n–simplex* in X. We write $\Delta_n(X)$ for the set of all singular n–simplices in X.

Remark 3.4. A singular simplex $\sigma \colon \Delta^n \longrightarrow X$ does not need to be an embedding; the image $\sigma(\Delta^n) \subset X$ may be degenerate.

Examples 3.5. Take $v_0, \ldots, v_n \in \mathbb{R}^N$. We denote by $[v_0, \ldots, v_n]$ the *affine singular n–simplex* in $\mathbb{R}^n$, i. e. $[v_0, \ldots, v_n] \colon \Delta^n \longrightarrow \mathbb{R}^N$ is the map such that

$$[v_0, \ldots, v_n]\Big(\sum t_i e_i\Big) = \sum t_i v_i.$$

The image of $[v_0, \ldots, v_n]$ is of course just the convex hull of $v_0, \ldots, v_n$. In particular, for $0 \leq i \leq n$ we have the affine $(n-1)$–simplex $[e_0, \ldots, \widehat{e_i}, \ldots, e_n] \colon \Delta^{n-1} \longrightarrow \Delta^n \subset \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$, the i^{th} *face* F_i^n of Δ^n. Here $\widehat{e_i}$ denotes the omission of e_i.

Definition 3.6. Let $C_n(X) = \mathbb{Z}^{(\Delta_n(X))}$ denote the free abelian group generated by $\Delta_n(X)$, i. e. $C_n(X)$ consists of formal linear combinations $\sum_i n_i \sigma_i$ of singular n–simplices $\sigma_i \in \Delta_n(X)$ with integral coefficients $n_i \in \mathbb{Z}$. The elements of $C_n(X)$ are called *singular n–chains* in X. The group homomorphism $\partial_n \colon C_n(X) \longrightarrow C_{n-1}(X)$ defined by

$$\partial_n \sigma = \sum_{i=0}^{n} (-1)^i \, \sigma \circ F_i^n.$$

for $\sigma \in \Delta_n(X)$ is called the *boundary operator.* The collection of all the $C_n(X)$ together with the ∂_n defines a *chain complex*

$$C_\bullet(X) = \ \cdots \longrightarrow C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0 \longrightarrow \cdots$$

called the *singular chain complex* of X. That $C_\bullet(X)$ is called a chain complex just means that $\partial_{n-1} \circ \partial_n = 0$ which we will see shortly.

Lemma 3.7. *We have $\partial_{n-1} \circ \partial_n = 0$ for all $n \in \mathbb{Z}$.*

Proof. Of course, it is enough to check this on $\Delta_n(X)$. For $\sigma \in \Delta_n(X)$ compute

$$\begin{aligned}\partial_{n-1}\partial_n\sigma = \partial_{n-1}\sum_{i=0}^{n}(-1)^i\,\sigma F_i^n = \sum_{i=0}^{n}(-1)^i\partial_{n-1}(\sigma F_i^n) = \sum_{i=0}^{n}(-1)^i\sum_{j=0}^{n-1}(-1)^j\,\sigma F_i^n F_j^{n-1} = \\ = \sum_{j<i}(-1)^{i+j}\,\sigma F_i^n F_j^{n-1} + \sum_{i\le j}(-1)^{i+j}\,\sigma F_i^n F_j^{n-1} = \\ = \sum_{j<i}(-1)^{i+j}\,\sigma F_i^n F_j^{n-1} + \sum_{i\le j}(-1)^{i+j}\,\sigma F_{j+1}^n F_i^{n-1} = \\ = \sum_{j<i}(-1)^{i+j}\,\sigma F_i^n F_j^{n-1} - \sum_{i<j}(-1)^{i+j}\,\sigma F_j^n F_i^{n-1} = 0.\end{aligned}$$

Here we have used

$$F_k^n \circ F_\ell^{n-1} = \begin{cases}[e_0,\dots,\widehat{e_\ell},\dots,\widehat{e_k},\dots,e_n], & \ell < k\\ [e_0,\dots,\widehat{e_k},\dots,\widehat{e_{\ell+1}},\dots,e_n], & \ell \ge k\end{cases}$$

which can be checked by direct calculation. □

We write $Z_n(X) \coloneqq \ker\partial_n \subset C_n(X)$ and $B_n(X) \coloneqq \operatorname{im}\partial_{n+1} \subset C_n(X)$. The elements of $Z_n(X)$ are called *singular n–cycles* and the elements of $B_n(X)$ are called *singular n–boundaries.* Lemma 3.7 implies that $B_n(X) \subset Z_n(X)$. Hence, the following definition makes sense.

Definition 3.8. The n^{th} *singular homology group* of X is the quotient

$$H_n(X) \coloneqq {Z_n(X)}/{B_n(X)}.$$

Remark 3.9.
- (i) The quotient $H_n(X)$ is an abelian group for all $n \in \mathbb{Z}$ by construction.
- (ii) Of course, $H_n(X) = 0$ for $n < 0$.
- (iii) Cycles $c_1, c_2 \in Z_n(X)$ are called *homologous* if and only if $c_2 - c_1 \in B_n(X)$, i.e. if $[c_1] = [c_2] \in H_n(X)$.

Definition 3.10. A family of abelian groups $(C_n)_{n\in\mathbb{Z}}$ together with group homomorphisms $\partial_n : C_n \longrightarrow C_{n-1}$ such that $\partial_{n-1} \circ \partial_n = 0$ is called a *chain complex.* We sometimes write

$$C_\bullet = \;\cdots \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow \cdots$$

for this collection of data. In general, the groups C_n are called *chain groups* and the maps ∂_n are called *boundary operators.* The quotient $H_n = Z_n/B_n$ where $Z_n = \ker\partial_n$ and $B_n = \operatorname{im}\partial_{n+1}$ is called the n^{th} *homology group* of $C_\bullet$.

A family $(\varphi_n\colon C_n \longrightarrow C'_n)_{n\in\mathbb{Z}}$ of homomorphisms between chain complexes $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ is called a *chain map* if the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & \\ \cdots & \longrightarrow & C'_n & \xrightarrow[\partial'_n]{} & C'_{n-1} & \longrightarrow & \cdots \end{array}$$

is commutative.

Lemma 3.11. *Any chain map $\varphi_\bullet\colon (C_\bullet, \partial_\bullet) \longrightarrow (C'_\bullet, \partial'_\bullet)$ induces natural group homomorphisms $\varphi_n\colon H_n(C_\bullet) \longrightarrow H_n(C'_\bullet)$.*

Proof. Let $c \in Z_n \subset C_n$. Then $\partial'_n\varphi_n(c) = \varphi_{n-1}\partial_n(c) = 0$. Hence, the φ_n restrict to homomorphisms $\varphi_n\colon Z_n \longrightarrow Z'_n$. If two cycles $c_1, c_2 \in Z_n$ differ only by a boundary, i. e. $c_2 - c_1 = \partial_{n+1}b$, then $\varphi_n(c_2) - \varphi_n(c_1) = \partial'_{n+1}\varphi_{n+1}(b)$. Hence, $\varphi_n(c_1) \equiv \varphi_n(c_2) \pmod{B'_n}$ and the φ_n descend to group homomorphisms $\varphi_n\colon H_n(C_\bullet) \longrightarrow H_n(C'_\bullet)$. □

Let $f\colon X \longrightarrow X'$ be a continuous map between topological spaces. Then we obtain natural maps $f_*\colon \Delta_n(X) \longrightarrow \Delta_n(X')$ via pre–composition, whence natural group homomorphisms $f_*\colon C_n(X) \longrightarrow C_n(X')$. These assemble to a chain map $f_*\colon C_\bullet(X) \longrightarrow C_\bullet(X')$ since

$$f_*\partial_n(\sigma) = \sum_{i=0}^{n}(-1)^i\ f \circ \sigma \circ F_i^n = \partial'_n f_*(\sigma)$$

for all $\sigma \in \Delta_n(X)$. We obtain natural homomorphisms $f_*\colon H_n(X) \longrightarrow H_n(X')$ on singular homology. It is immediate that $(fg)_* = f_*g_*$ and $\mathrm{id}_* = \mathrm{id}$ and we obtain a family of functors $H_n\colon \mathbf{Top} \longrightarrow \mathbf{Ab}$. Hence, if X and X' are homeomorphic then $H_n(X) \cong H_n(X')$ for all $n \in \mathbb{Z}$. The converse need not hold!

Examples 3.12. We will compute the homology of the one point space $*$. It has a unique singular simplex $\sigma_n\colon \Delta_n \longrightarrow *$ for all $n \geq 0$, i. e. $C_n(X) = \mathbb{Z}\sigma_n$ for $n \geq 0$. The boundary of σ_n is

$$\partial_n\sigma = \sum_{i=0}^{n}(-1)^i\sigma_{n-1} = \begin{cases} 0, & n \text{ odd} \\ \sigma_{n-1}, & n \text{ even} \end{cases}$$

for $n \geq 1$ and $\partial_n = 0$ for $n < 1$. Hence, $\partial_n\colon C_n(*) \longrightarrow C_{n-1}(*)$ is an isomorphism if $n \geq 0$ is even and $\partial_n = 0$ otherwise. The chain complex $C_\bullet(*)$ looks like

$$\cdots \xrightarrow{0} \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

We obtain the singular homology as

$$H_n(*) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0. \end{cases}$$

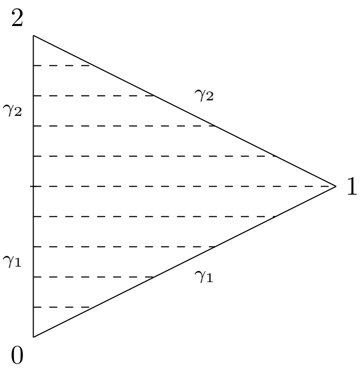


Figure 6: The 2–simplex σ.

Proposition 3.13. *If $X = \bigsqcup_{i\in I} X_i$ is the decomposition of X into its path components, then*

$$H_n(X) \cong \bigoplus_{i\in I} H_n(X_i)$$

for all $n \in \mathbb{Z}$.

Proof. Because the standard n–simplex $\Delta^n \subset \mathbb{R}^{n+1}$ is path–connected, its image $\sigma(\Delta^n)$ under any singular simplex $\sigma \in \Delta_n(X)$ is entirely contained in one of the X_i. This implies that

$$C_n(X) \cong \bigoplus_{i\in I} C_n(X_i)$$

for all $n \in \mathbb{N}$. For $\sigma \in \Delta_n(X_i)$ clearly $\partial_n\sigma \in \Delta_{n-1}(X_i)$. Hence, $\partial_n^X = \sum_i \partial_n^{X_i}$ and the result follows. □

3.2 H_0 and H_1

Consider the *augmentation* homomorphism $\varepsilon\colon C_0(X) \longrightarrow \mathbb{Z}$ satisfying $\varepsilon(x) = 1$ for $x \in X$; here we have identified a 0–simplex $\Delta_0 \longrightarrow X$ with its image. Then $\varepsilon \circ \partial_1 = 0$. Conversely, if X is path–connected and non–empty then $\varepsilon(c) = 0$ implies that $c \in B_0(X)$: If $c = \sum_{i=0}^k n_i x_i$, choose a base point $x \in X$ and paths σ_i from x to x_i for $i = 0, \dots, k$ and consider the σ_i as 1–simplices in X. Then $\partial\sigma_i = x_i - x$ and

$$\partial \sum_{i=0}^k n_i\sigma_i = \sum_{i=0}^k n_i x_i - \varepsilon(c)x = c,$$

i. e. $c \in B_0(X)$. In summary, if $X \neq \emptyset$ is path–connected, then $\ker\varepsilon = B_0(X)$ and the augmentation factors through $Z_0(X) \twoheadrightarrow H_0(X)$ and yields an isomorphism $H_0(X) \xrightarrow{\sim} \mathbb{Z}$. Hence, for any space X we have $H_0(X) = \mathbb{Z}^{(\pi_0(X))}$.

Suppose now that (X, x_0) is a path–connected pointed space and take any loop $\gamma\colon I \longrightarrow X$ in x_0 considered as a 1–chain in X. Observe that $\partial\gamma = 0$, i. e. $\gamma \in Z_1(X)$. If γ' and γ are homotopic loops in x_0 then $\gamma - \gamma' \in B_1(X)$: Given a homotopy $H\colon I \times I \longrightarrow X$ from γ' to

γ fixing the endpoints, collapsing $\{0\} \times I$ to a point gives a singular 2–simplex $\sigma\colon \Delta^2 \longrightarrow X$. Its boundary is given by $\partial\sigma = \gamma + c_{x_0} - \gamma' = \gamma - \gamma' + \partial\sigma'$ where σ' is the constant 2–simplex in x_0, hence $\gamma - \gamma' \in B_1(X)$. We obtain a homomorphism $\phi\colon \pi_1(X, x_0) \longrightarrow H_1(X)$ because $\gamma_1 * \gamma_2 = \gamma_1 + \gamma_2 - \partial\sigma$ with $\sigma \in \Delta_2(X)$ given by $\sigma \circ [e_0, e_1] = \gamma_1$, $\sigma \circ [e_1, e_2] = \gamma_2$, $\sigma \circ [e_0, e_2] = \gamma_1 * \gamma_2$ and extending constantly along lines perpendicular to $[e_0, e_2]$—see Figure 6. Because $H_1(X)$ is abelian the homomorphism ϕ factors uniquely as

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\quad\phi\quad} & H_1(X) \\ & \searrow \quad \nearrow{\scriptstyle\Phi} & \\ & \pi_1(X, x_0)^{\mathrm{ab}} & \end{array}$$

where $\pi_1(X, x_0)^{\mathrm{ab}}$ denotes the *abelianisation* of $\pi_1(X, x_0)$, i. e. the biggest abelian quotient of $\pi_1(X, x_0)$. The homomorphism Φ is called the *Hurewicz homomorphism.*

Proposition 3.14 (Hurewicz)**.** *For any path–connected pointed space (X, x_0) the Hurewicz homomorphism is an isomorphism.*

Proof. We construct an inverse $\Psi\colon H_1(X) \longrightarrow \pi_1(X, x_0)^{\mathrm{ab}}$ as follows. For all $x \in X$ choose a path τ_x from x_0 to x. For $\gamma \in \Delta_1(X)$ set

$$\Psi(\gamma) = [\tau_{\gamma(0)} * \gamma * \tau^-_{\gamma(1)}] \in \pi_1(X, x_0)^{\mathrm{ab}}.$$

By linear extension we obtain a homomorphism $\Psi\colon C_1(X) \longrightarrow \pi_1(X, x_0)^{\mathrm{ab}}$ which satisfies $\Psi(B_1(X)) = 0$: Indeed, for $\sigma \in \Delta_2(X)$ write $\gamma_i = \sigma \circ F^n_i$ and $y_i = \sigma(e_i)$ and observe that

$$\begin{aligned} \Psi(\partial\sigma) = \Psi(\gamma_0 - \gamma_1 + \gamma_2) &= [\tau_{y_1} * \gamma_0 * \tau^-_{y_2}] - [\tau_{y_0} * \gamma_1 * \tau^-_{y_2}] + [\tau_{y_0} * \gamma_2 * \tau^-_{y_1}] = \\ &= [\tau_{y_1} * \gamma_0 * \tau^-_{y_2} * \tau_{y_2} * \gamma_1^- * \tau^-_{y_0} * \tau_{y_0} * \gamma_2 * \tau^-_{y_1}] = \\ &= [\tau_{y_1} * \gamma_0 * \gamma_1^- * \gamma_2 * \tau^-_{y_1}] = 0 \end{aligned}$$

because $\gamma_0 * \gamma_1^- * \gamma_2$ is null–homotopic. Hence, the homomorphism Ψ descends to a homomorphism $H_1(X) \longrightarrow \pi_1(X, x_0)^{\mathrm{ab}}$.

Now, for a loop γ in x_0 one has $\Psi(\Phi([\gamma])) = [\tau_{x_0} * \gamma * \tau^-_{x_0}] = [\gamma] \in \pi_1(X, x_0)^{\mathrm{ab}}$. Conversely, given any singular 1–simplex γ in X we have

$$\Phi(\Psi([\gamma])) = [\tau_{\gamma(0)} * \gamma * \tau^-_{\gamma(1)}].$$

As in the case of loops based at x_0, we have $\gamma_1 * \gamma_2 = \gamma_1 + \gamma_2 + \partial\sigma$ for arbitrary paths γ_i in X and some 2–simplex σ. Hence, $\Phi(\Psi([\gamma])) = [\tau_{\gamma(0)}] + [\gamma] - [\tau_{\gamma(1)}]$. Now, assume that $C = \sum_i a_i\gamma_i \in Z_1(X)$, i. e. $\partial C = \sum_i a_i(\gamma_i(1) - \gamma_i(0)) = 0$. Then

$$\Phi(\Psi([C])) = \left[\sum_i a_i\gamma_i + \sum_i a_i(\tau_{\gamma_i(0)} - \tau_{\gamma_i(1)})\right] = [C]. \qquad \square$$

3.3 Homotopy Invariance

Let $C_\bullet$ and $C'_\bullet$ be chain complexes together with chain maps $\Phi, \Psi\colon C_\bullet \longrightarrow C'_\bullet$.

Definition 3.15. A *chain homotopy* h from Φ to Ψ is a family of group homomorphisms $h_n \colon C_n \longrightarrow C'_{n+1}$ such that $h_{n-1}\partial_n + \partial'_{n+1}h_n = \Psi_n - \Phi_n$ for all $n \in \mathbb{Z}$.

Lemma 3.16. *Chain maps $\Phi, \Psi \colon C_\bullet \longrightarrow C'_\bullet$ admitting a chain homotopy h from Φ to Ψ satisfy $H_n(\Phi) = H_n(\Psi)$ for all $n \in \mathbb{Z}$.*

Proof. Given any cycle $c \in Z_n(C_\bullet) = \ker(\partial_n)$ we have

$$\Psi_n(c) - \Phi_n(c) = h_{n-1}(\partial_n c) + \partial'_{n+1}h_n(c) \equiv 0 \pmod{B_n(C'_\bullet)}$$

and hence $H_n(\Phi) = H_n(\Psi)$. □

Theorem 3.17. *Consider continuous maps $f, g \colon X \longrightarrow X'$ between topological spaces X and Y. Let $H \colon X \times I \longrightarrow X'$ be a homotopy from f to g. Then H induces a chain homotopy between from f_* and g_* as chain maps $C_\bullet(X) \longrightarrow C_\bullet(X')$ between the singular complexes. Hence, $f_* = g_* \colon H_\bullet(X) \longrightarrow H_\bullet(X')$ on homology.*

Proof. Take any $\sigma \colon \Delta^n \longrightarrow X$. The homotopy H yields a map $H_\sigma = H \circ (\sigma \times \mathrm{id}_I) \colon \Delta^n \times I \longrightarrow X'$. Then $H_\sigma(\_, 0) = f \circ \sigma = f_*(\sigma)$ and $H_\sigma(\_, 1) = g_*(\sigma)$. Now, the idea is to turn H_σ into a $(n+1)$–chain of X' by subdividing $\Delta^n \times I$ into simplices.Consider $\Delta^n \times I$ as a subset of $\mathbb{R}^{n+1} \times \mathbb{R} = \mathbb{R}^{n+2}$ identifying $e_i \in \Delta^n$ with $(e_i, 0) \in \mathbb{R}^{n+2}$ and set $f_i = (e_i, 1) \in \mathbb{R}^{n+2}$. The affine $(n+1)$–simplices $[e_0, \ldots, e_i, f_i, \ldots, f_n] \colon \Delta^{n+1} \longrightarrow \Delta^n \times I$ form a decomposition of $\Delta^n \times I$, i. e.

$$\Delta^n \times I = \bigcup_{i=0}^{n} [e_0, \ldots, e_i, f_i, \ldots, f_n](\Delta^{n+1}),$$

and

$$[e_0, \ldots, e_i, f_i, \ldots, f_n](\Delta^{n+1}) \cap [e_0, \ldots, e_{i+1}, f_{i+1}, \ldots, f_n](\Delta^{n+1}) = [e_0, \ldots, e_i, f_{i+1}, \ldots, f_n](\Delta^n).$$

Define $h_n \colon C_n(X) \longrightarrow C_{n+1}(X')$ by

$$h_n(\sigma) = \sum_{i=0}^{n} (-1)^i \; H_\sigma \circ [e_0, \ldots, e_i, f_i, \ldots, f_n]$$

for $\sigma \in \Delta_n(X)$.

Compute:

$$\begin{aligned}
\partial' h_n(\sigma) &= \partial' \sum_{i=0}^{n} (-1)^i \; H_\sigma \circ [e_0, \ldots, e_i, f_i, \ldots, f_n] = \\
&= \sum_{j \le i} (-1)^{i+j} \; H_\sigma \circ [e_0, \ldots, \widehat{e_j}, \ldots, e_i, f_i, \ldots, f_n] + \\
&\quad + \sum_{j \ge i} (-1)^{i+j+1} \; H_\sigma \circ [e_0, \ldots, e_i, f_i, \ldots, \widehat{f_j}, \ldots, f_n] = \\
&= H_\sigma \circ [\widehat{e_0}, f_0, \ldots, f_n] - H_\sigma \circ [e_0, \ldots, e_n, \widehat{f_n}] + \\
&\quad + \sum_{j < i} (-1)^{i+j} \; H_\sigma \circ [e_0, \ldots, \widehat{e_j}, \ldots, e_i, f_i, \ldots, f_n] + \\
&\quad + \sum_{i < j} (-1)^{i+j+1} \; H_\sigma \circ [e_0, \ldots, e_i, f_i, \ldots, \widehat{f_j}, \ldots, f_n]
\end{aligned}$$

and

$$h_{n-1}(\partial\sigma) = h_{n-1}\sum_{i=0}^{n}(-1)^i\ \sigma\circ[e_0,\dots,\widehat{e_i},\dots,e_n] =$$
$$= \sum_{j<i}(-1)^{i+j}\ H_\sigma\circ[e_0,\dots,e_j,f_j,\dots,\widehat{f_i},\dots,f_n] +$$
$$+\sum_{i<j}(-1)^{i+j+1}H_\sigma\circ[e_0,\dots,\widehat{e_i},\dots,e_j,f_j,\dots,f_n].$$

Hence, $\partial' h_n(\sigma) + h_{n+1}(\partial\sigma) = g_*(\sigma) - f_*(\sigma)$. □

Corollary 3.18. *Any homotopy equivalence $f\colon X \longrightarrow X'$ between topological spaces X and X' induces isomorphisms $f_*\colon H_n(X) \xrightarrow{\sim} H_n(X')$ for all $n \in \mathbb{Z}$.*

3.4 Long Exact Homology Sequence & Excision

Given X and $A \subset X$, can we compute $H_\bullet(X)$ from $H_\bullet(A)$ and $H_\bullet(X/A)$? We will first replace $H_\bullet(X/A)$ by the "relative homology" $H_\bullet(X,A)$.

Definition 3.19. For a pair (X,A), that is a topological space X with a subspace $A \subset X$, consider $C_n(X,A) = C_n(X)/C_n(A)$ for all $n \in \mathbb{Z}$. There is an induced boundary operator $\partial_n\colon C_n(X,A) \longrightarrow C_{n-1}(X,A)$ and the homology $H_\bullet(X,A)$ of the corresponding complex is called the *relative homology* of the pair (X,A).

Definition 3.20. Given pairs (X,A) and (X',A') of topological spaces, a continuous map $f\colon X \longrightarrow X'$ is called a *map of pairs* if $f(A) \subset A'$. Any map of pairs $f\colon (X,A) \longrightarrow (X',A')$ induces a group homomorphism $f_*\colon H_\bullet(X,A) \longrightarrow H_\bullet(X',A')$ on relative homology.

Consider the inclusions $i\colon A \hookrightarrow X$ and $j\colon (X,\emptyset) \longrightarrow (X,A)$ with induced homomorphisms $i_*\colon H_\bullet(A) \longrightarrow H_\bullet(X)$ and $j_*\colon H_\bullet(X) \longrightarrow H_\bullet(X,A)$. Now, it is immediate that there is a short exact sequence

$$0 \longrightarrow C_\bullet(A) \xrightarrow{i_*} C_\bullet(X) \xrightarrow{j_*} C_\bullet(X,A) \longrightarrow 0$$

of complexes.

Proposition 3.21. *Any short exact sequence*

$$0 \longrightarrow A_\bullet \xrightarrow{\alpha} B_\bullet \xrightarrow{\beta} C_\bullet \longrightarrow 0$$

induces a long exact sequence

$$\cdots \longrightarrow H_{n+1}(C_\bullet) \xrightarrow{\delta_{n+1}} H_n(A_\bullet) \xrightarrow{H_n(\alpha)} H_n(B_\bullet) \xrightarrow{H_n(\beta)} H_n(C_\bullet) \xrightarrow{\delta_n} H_{n-1}(A_\bullet) \longrightarrow \cdots$$

where δ_n is given by the formula $\delta_n([c]) = [\alpha_{n-1}^{-1}\partial\beta_n^{-1}(c)]$. Furthermore, δ_n is natural in the sense that for any commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_\bullet & \xrightarrow{\alpha} & B_\bullet & \xrightarrow{\beta} & C_\bullet & \longrightarrow & 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h & & \\ 0 & \longrightarrow & A'_\bullet & \xrightarrow{\alpha'} & B'_\bullet & \xrightarrow{\beta'} & C'_\bullet & \longrightarrow & 0 \end{array}$$

the induced diagram

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_{n+1}(C_\bullet) & \xrightarrow{\delta_{n+1}} & H_n(A_\bullet) & \xrightarrow{H_n(\alpha)} & H_n(B_\bullet) & \xrightarrow{H_n(\beta)} & H_n(C_\bullet) & \xrightarrow{\delta_n} & H_{n-1}(A_\bullet) & \longrightarrow & \cdots \\ & & \downarrow H_{n+1}(h) & & \downarrow H_n(f) & & \downarrow H_n(g) & & \downarrow H_n(h) & & \downarrow H_{n-1}(f) & & \\ \cdots & \longrightarrow & H_{n+1}(C'_\bullet) & \xrightarrow[\delta'_{n+1}]{} & H_n(A'_\bullet) & \xrightarrow[H_n(\alpha')]{} & H_n(B'_\bullet) & \xrightarrow[H_n(\beta')]{} & H_n(C'_\bullet) & \xrightarrow[\delta'_n]{} & H_{n-1}(A'_\bullet) & \longrightarrow & \cdots \end{array}$$

is also commutative.

Proof. To quote Charles Weibel:

> We will not print the proof in these notes, because it is best done visually. In fact, a clear proof is given by Jill Clayburgh at the beginning of the movie *It's My Turn* (Rastar–Martin Elfand Studios, 1980). As an exercise in "diagram chasing" of elements, the student should find a proof (but privately—keep the proof to yourself!).

We obtain a long exact sequence

$$\cdots \longrightarrow H_{n+1}(X,A) \xrightarrow{\delta_{n+1}} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\delta_n} H_{n-1}(A) \longrightarrow \cdots$$

called the *long exact sequence of the pair* (X,A). There is a geometric description of the connecting homomorphisms δ_n. A homology class $\gamma \in H_n(X,A)$ is represented by a relative cycle, i. e. a chain $b \in C_n(X)$ such that $\partial b \in C_n(A)$. Then $\delta_n(\gamma)$ is just the homology class $[\partial_n(b)] \in H_n(A)$.

Let (X,A) be a pair and $B \subset A$ such that $\overline{B} \subset A^\circ$. We wish to show that the inclusion $i\colon (X \smallsetminus B, A \smallsetminus B) \hookrightarrow (X,A)$ induces isomorphisms on homology. To this end we will try to invert $i_*\colon C_\bullet(X \smallsetminus B, A \smallsetminus B) \longrightarrow C_\bullet(X,A)$—which of course won't work literally. Consider the open cover $\mathscr{U} = \{A^\circ, X \smallsetminus \overline{B}\}$ of X. *If* for a relative cycle $c = \sum_i n_i\sigma_i \in C_n(X)$ one always has $\sigma_i \in \Delta_n(A^\circ)$ or $\sigma_i \in \Delta_n(X \smallsetminus \overline{B})$, *then* by throwing away the summands of c with $\sigma_i \in \Delta_n(A^\circ)$ one would obtain a chain which is disjoint from $\overline{B}$ and which is equivalent to c modulo $C_n(A)$. But in general this will not be possible.

Instead, we will construct a chain map $\phi\colon C_\bullet(X) \longrightarrow C_\bullet(X)$ together with a chain homotopy $h\colon C_\bullet(X) \longrightarrow C_{\bullet+1}(X)$ from ϕ to id such that for an open cover $\mathscr{U} = (U_i)_{i\in I}$ of X and $\sigma \in \Delta_n(X)$ there exists a $k \in \mathbb{N}$ such that any simplex of $\phi^k(\sigma) \in C_n(X)$ lies completely in one of the U_i. There is a systematic way of doing this called *barycentric subdivision.*

Consider $\Delta^n \subset \mathbb{R}^{n+1}$ and the subcomplex $L_\bullet(\Delta^n) \subset C_\bullet(\Delta^n)$ generated by the affine singular simplices $[v_0,\dots,v_p]\colon \Delta^p \longrightarrow \Delta^n$ with $v_i \in \Delta^n$. For $\sigma = [v_0,\dots,v_p]$ and $v \in \Delta^n$ consider

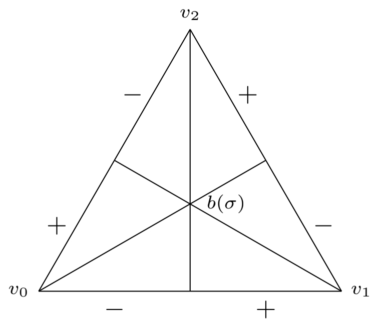


Figure 7: Barycentric subdivision

the *cone* $c_v(\sigma) = [v, v_0, \dots, v_p]$ on σ with vertex v. This extends linearly to a homomorphism $c_v \colon L_p(\Delta^n) \longrightarrow L_{p+1}(\Delta^n)$. We have

$$\partial c_v([v_0, \dots, v_p]) = [v_0, \dots, v_p] + \sum_{i=0}^{p+1} (-1)^{i+1} [v, v_0, \dots, v_i, \dots, v_p] = \sigma - c_v(\partial\sigma)$$

for $p > 0$ and

$$\partial c_v(\sigma) = \partial[v, \sigma(e_0)] = \sigma - [v]$$

for $p = 0$. For a chain $c \in L_p(\Delta^n)$ we have similarly

$$\partial c_v(\sigma) = \begin{cases} c - c_v(\partial\sigma) & p > 0 \\ c - \varepsilon(c)[v] & p = 0. \end{cases}$$

Definition 3.22. Define $\phi\colon L_\bullet(\Delta^n) \longrightarrow L_\bullet(\Delta^n)$ inductively as follows:

$$\phi(\sigma) = \begin{cases} c_{b(\sigma)}(\phi(\partial\sigma)) & p > 0 \\ \sigma & p = 0, \end{cases}$$

where $\sigma = [v_0, \dots, v_p]$ and

$$b([v_0, \dots, v_p) = \frac{1}{p+1} \sum_{i=0}^{p} v_i$$

is the *barycenter* of $[v_0, \dots, v_p]$. This extends linearly to chains.

Examples 3.23. For $\sigma = [v_0, v_1]$ we have $\partial\sigma = [v_1] - [v_0]$ and

$$\phi(\sigma) = [(v_0 + v_1)/2, v_1] - [(v_0 + v_1)/2, v_0].$$

For $\sigma = [v_0, v_1, v_2]$ we have $\partial\sigma = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$ and

$$\begin{aligned} \phi(\partial\sigma) = {} & [(v_1 + v_2)/2, v_2] - [(v_1 + v_2)/2, v_1] - [(v_0 + v_2)/2, v_2] + [(v_0 + v_2)/2, v_0] + \\ & + [(v_0 + v_1)/2, v_1] - [(v_0 + v_1)/2, v_0]. \end{aligned}$$

Hence,

$$\phi(\sigma) = [b, (v_1 + v_2)/2, v_2] - [b, (v_1 + v_2)/2, v_1] - [b, (v_0 + v_2)/2, v_2] + [b, (v_0 + v_2)/2, v_0] + \\ + [b, (v_0 + v_1)/2, v_1] - [b, (v_0 + v_1)/2, v_0].$$

where $b = (v_0 + v_1 + v_2)/3$.

Lemma 3.24. *The homomorphism $\phi\colon L_\bullet(\Delta^n) \longrightarrow L_\bullet(\Delta^n)$ is a chain map.*

Proof. Let $\sigma = [v_0, \dots, v_p]$ be an affine p–simplex in Δ^n. For $p = 0$, we have $\phi(\partial\sigma) = \partial\phi(\sigma)$ trivially. For $p > 0$ we have

$$\partial\phi(\sigma) = \phi(\partial\sigma) - c_{b(\sigma)}(\partial\phi(\partial\sigma)) = \\ = \phi(\partial\sigma) - c_{b(\sigma)}(\phi(\partial^2\sigma)) = \phi(\partial\sigma).$$

□

Define a chain map $h\colon L_\bullet(\Delta^n) \longrightarrow L_{\bullet+1}(\Delta^n)$ inductively by

$$h(\sigma) = \begin{cases} 0 & p = 0 \\ c_{b(\sigma)}(\phi(\sigma) - \sigma - h(\partial\sigma)) & p > 0 \end{cases}$$

on p–simplices σ and extend linearly to chains.

Lemma 3.25. *The map h is a chain homotopy from ϕ to* id*, i. e. $\partial \circ h + h \circ \partial = \phi - \mathrm{id}$.*

Proof. For $p = 0$ we trivially have $\partial h(\sigma) + h(\partial\sigma) = 0 = \phi(\sigma) - \sigma$. For $p > 0$ we have

$$\begin{aligned}\partial h(\sigma) &= \partial c_{b(\sigma)}(\phi(\sigma) - \sigma - h(\partial\sigma)) = \\ &= \phi(\sigma) - \sigma - h(\partial\sigma) - c_{b(\sigma)}(\partial\phi(\sigma) - \partial\sigma - \partial h(\partial\sigma)) = \\ &= \phi(\sigma) - \sigma - h(\partial\sigma) - c_{b(\sigma)}(\partial\phi(\sigma) - \partial\sigma - \phi(\partial\sigma) + \partial\sigma) = \\ &= \phi(\sigma) - \sigma - h(\partial\sigma).\end{aligned}$$

□

Coming back to the general situation, define $\phi\colon C_\bullet(X) \longrightarrow C_\bullet(X)$ by $\phi(\sigma) = \sigma_*(\phi(\mathrm{id}_{\Delta^n}))$ for $\sigma \in \Delta^n(X)$ and linear extension. Similarly, define a chain homotopy $h\colon C_\bullet(X) \longrightarrow C_{\bullet+1}(X)$ by $h(\sigma) = \sigma_*(h(\mathrm{id}_{\Delta^n}))$ for $\sigma \in \Delta_n(X)$. It can be checked that this is compatible with the previous definitions for $X = \Delta^n$ on $L_\bullet(\Delta^n)$.

Lemma 3.26. *Defined this way, we obtain a chain map $\phi\colon C_\bullet(X) \longrightarrow C_\bullet(X)$ which is chain homotopic to* id *via h. Consequently, for any $k \geq 1$ the iterate ϕ^k is chain homotopic to* id*.*

Proof. First note that for $f\colon X \longrightarrow X'$ we have $\phi \circ f_* = f_* \circ \phi$ and $h \circ f_* = f_* \circ h$. For $\sigma \in \Delta_n(X)$ it follows that

$$\phi(\partial\sigma) = \phi(\partial\sigma_*(\mathrm{id}_{\Delta^n})) = \phi(\sigma_*(\partial\,\mathrm{id}_{\Delta^n})) = \sigma_*(\phi(\partial\,\mathrm{id}_{\Delta^n})) = \\ = \sigma_*(\partial\phi(\mathrm{id}_{\Delta^n})) = \partial\sigma_*(\phi(\mathrm{id}_{\Delta^n})) = \partial\phi(\sigma)$$

by Lemma 3.24.

Additionally,

$$\partial h(\sigma) = \partial\sigma_*(h(\mathrm{id}_{\Delta^n})) = \sigma_*(\partial h(\mathrm{id}_{\Delta^n}))$$

and

$$h(\partial\sigma) = h(\partial\sigma_*(\mathrm{id}_{\Delta^n})) = \sigma_*(h(\partial\,\mathrm{id}_{\Delta^n}))$$

which implies

$$\partial h(\sigma) + h(\partial\sigma) = \sigma_*(\phi(\mathrm{id}_{\Delta^n}) - \mathrm{id}_{\Delta^n}) = \phi(\sigma) - \sigma$$

because of Lemma 3.25. □

Lemma 3.27. *For $\sigma = [v_0, \ldots, v_p]\colon \Delta^p \longrightarrow \Delta^n$ the diameter of any simplex appearing in $\phi(\sigma)$ is at most*

$$\frac{p}{p+1}\,\mathrm{diam}(\sigma(\Delta^p)).$$

Consequently, any simplex appearing in $\phi^k(\sigma)$ has diameter at most

$$\left(\frac{p}{p+1}\right)^k \mathrm{diam}(\sigma(\Delta^p)).$$

Proof. Each simplex σ' appearing in $\phi(\sigma)$ has the form

$$\sigma' = \left[\frac{1}{p+1}\sum_{i=0}^{p} v_i, \ldots, \frac{1}{3}(v_{i_2} + v_{i_1} + v_{i_0}), \frac{1}{2}(v_{i_1} + v_{i_0}), v_{i_0}\right] =: [v'_0, \ldots, v'_{p-2}, v'_{p-1}, v'_p].$$

Furthermore, $\mathrm{diam}\,\sigma(\Delta^p) = \max \|v_i - v_j\|$ and $\mathrm{diam}\,\sigma'(\Delta^p) = \max \|v'_i - v'_j\|$. The claim then follows from the following general statement: For vectors $w_0, \ldots, w_p \in \mathbb{R}^{n+1}$ and $0 \leq \ell < k \leq p$ one has the estimate

$$\left\|\frac{1}{k+1}\sum_{i=0}^{k} w_i - \frac{1}{\ell+1}\sum_{i=0}^{\ell} w_i\right\| \leq \frac{p}{p+1}\max_{0\leq i,j\leq p}\|w_i - w_j\|.$$

This follows from the following computation:

$$\begin{aligned}
\left\|\frac{1}{k+1}\sum_{i=0}^{k} w_i - \frac{1}{\ell+1}\sum_{i=0}^{\ell} w_i\right\| &= \left\|\frac{1}{k+1}\sum_{i=0}^{\ell} w_i - \frac{1}{\ell+1}\sum_{i=0}^{\ell} w_i + \frac{1}{k+1}\sum_{i=\ell+1}^{k} w_i\right\| = \\
&= \left\|\frac{\ell-k}{(k+1)(\ell+1)}\sum_{i=0}^{\ell} w_i + \frac{1}{k+1}\sum_{i=\ell+1}^{k} w_i\right\| = \\
&= \frac{k-\ell}{k+1}\left\|\frac{1}{\ell+1}\sum_{i=0}^{\ell} w_i - \frac{1}{k-\ell}\sum_{i=\ell+1}^{k} w_i\right\| \leq \\
&\leq \frac{k}{k+1}\max_{0\leq i,j\leq p}\|w_i - w_j\| \leq \frac{p}{p+1}\max_{0\leq i,j\leq p}\|w_i - w_j\|.
\end{aligned}$$

□

Definition 3.28. Let $\mathscr{U} = (U_i)_{i\in I}$ be a cover of a topological space X such that $X = \bigcup_i U_i^\circ$. A singular chain $c = \sum_j a_j\sigma_j \in C_p(X)$ is called *$\mathscr{U}$–small* if for each j there is some $i \in I$ such that $\sigma_j(\Delta^p) \subset U_i$.

Exercise 3.29 (Lebesgue Lemma)**.** Let X be a compact metric space together with an open cover $\mathscr{U} = (U_i)_{i\in I}$ of X. Then there exists a $\delta > 0$ such that any closed subset $A \subset X$ with $\mathrm{diam}(A) < \delta$ is completely contained in some U_i.

Lemma 3.30. *Let $\mathscr{U} = (U_i)_{i\in I}$ be a cover of a topological space X such that $X = \bigcup_i U_i^\circ$. For each $\sigma \in \Delta_n(X)$ there exists a $k \geq 0$ such that $\phi^k(\sigma)$ is $\mathscr{U}$–small.*

Proof. Consider the induced open cover $(\sigma^{-1}(U_i^\circ))_{i\in I}$ of Δ^n and then apply Lemma 3.27 and Exercise 3.29. ☐

Definition 3.31. Let $C_\bullet^{\mathscr{U}}(X)$ be the subcomplex of $C_\bullet(X)$ consisting of $\mathscr{U}$–small chains. Write $H_n^{\mathscr{U}}(X) = H_n(C_\bullet^{\mathscr{U}}(X))$ for $n \in \mathbb{Z}$.

Proposition 3.32. *The map $H_n^{\mathscr{U}}(X) \longrightarrow H_n(X)$ induced by the inclusion $C_\bullet^{\mathscr{U}}(X) \hookrightarrow C_\bullet(X)$ is an isomorphism for all $n \in \mathbb{Z}$.*

Proof. To prove surjectivity consider any $[c] \in H_n(X)$. Then there exists some $c' \in C_n^{\mathscr{U}}(X)$ such that $\partial c' = 0$ and $[c'] = [c] \in H_n(X)$: Indeed, Lemma 3.30 implies that $c' = \phi^k(c)$ is $\mathscr{U}$–small for $k \gg 0$. Furthermore ϕ^k is chain homotopic to id, that is $c' - c = h_k(\partial c) + \partial h_k(c)$ for some chain homotopy h_k. Hence, $\partial c' = 0$ and $c' - c \equiv 0 \pmod{B_n(X)}$, i. e. $[c'] = [c] \in H_n(X)$.

For injectivity consider any $[c] \in H_n^{\mathscr{U}}(X)$ such that $[c] = 0 \in H_n(X)$, i. e. $c = \partial b$ for some $b \in C_{n+1}(X)$. Lemma 3.30 implies that $\phi^k(b)$ is $\mathscr{U}$–small for $k \gg 0$ and we again have $\phi^k(b) - b = h_k(c) + \partial h_k(b)$ and $\partial\phi^k(b) - c = \partial h_k(c) \equiv 0 \pmod{B_n^{\mathscr{U}}(X)}$ because c and $h_k(c)$ are $\mathscr{U}$–small. This implies $[c] = 0 \in H_n^{\mathscr{U}}(X)$. ☐

For a pair (X, A) of spaces and a cover $\mathscr{U} = (U_i)_{i\in I}$ of X we write $\mathscr{U} \cap A = (U_i \cap A)_{i\in I}$ for the induced cover of A. Also, define the relative singular complex $C_\bullet^{\mathscr{U}}(X, A) := C_\bullet^{\mathscr{U}}(X)/C_\bullet^{\mathscr{U}\cap A}(A)$ and the relative homology $H_n^{\mathscr{U}}(X, A) := H_n(C_\bullet^{\mathscr{U}}(X, A))$. We obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_\bullet^{\mathscr{U}}(A) & \xrightarrow{i_*} & C_\bullet^{\mathscr{U}}(X) & \xrightarrow{j_*} & C_\bullet^{\mathscr{U}}(X,A) & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & C_\bullet(A) & \xrightarrow{i_*} & C_\bullet(X) & \xrightarrow{j_*} & C_\bullet(X,A) & \longrightarrow & 0 \end{array}$$

of chain complexes with exact rows and hence a commutative diagram

$$\begin{array}{ccccccccccccc} \cdots & \longrightarrow & H_{n+1}^{\mathscr{U}}(X,A) & \xrightarrow{\partial_{n+1}} & H_n^{\mathscr{U}}(A) & \xrightarrow{i_*} & H_n^{\mathscr{U}}(X) & \xrightarrow{j_*} & H_n^{\mathscr{U}}(X,A) & \xrightarrow{\partial_n} & H_{n-1}^{\mathscr{U}}(A) & \longrightarrow & \cdots \\ & & \big\downarrow & & \big\downarrow\cong & & \big\downarrow\cong & & \big\downarrow & & \big\downarrow\cong & & \\ \cdots & \longrightarrow & H_{n+1}(X,A) & \xrightarrow[\partial_{n+1}]{} & H_n(A) & \xrightarrow[i_*]{} & H_n(X) & \xrightarrow[j_*]{} & H_n(X,A) & \xrightarrow[\partial_n]{} & H_{n-1}(A) & \longrightarrow & \cdots \end{array}$$

in homology with exact rows. Hence, the five lemma implies that the inclusion $C_\bullet^{\mathscr{U}}(X) \hookrightarrow C_\bullet(X)$ also induces isomorphisms $H_n^{\mathscr{U}}(X, A) \xrightarrow{\sim} H_n(X, A)$.

Theorem 3.33 (Excision)**.** *Let (X, A) be a pair of spaces and assume $B \subset A$ is a subspace such that $\overline{B} \subset A^\circ$. Then the inclusion $i\colon (X \setminus B, A \setminus B) \hookrightarrow (X, A)$ induces isomorphisms*

$$i_*\colon H_n(X \setminus B, A \setminus B) \xrightarrow{\sim} H_n(X, A)$$

for all $n \in \mathbb{Z}$.

Proof. Consider the cover $\mathscr{U} = \{X \setminus B, A\}$ of X. Then $X = (X \setminus B)^\circ \cup A^\circ$ and in particular there is an isomorphism $H_n^{\mathscr{U}}(X, A) \xrightarrow{\sim} H_n(X, A)$. Now,

$$C_\bullet^{\mathscr{U}}(X) = C_\bullet(X \setminus B) + C_\bullet(A)$$

and

$$C_\bullet^{\mathscr{U} \cap A}(A) = C_\bullet(A),$$

hence

$$C_\bullet^{\mathscr{U}}(X, A) = {C_\bullet(X \setminus B) + C_\bullet(A)}\big/{C_\bullet(A)}.$$

We factor the inclusion on the chain level,

$$\begin{array}{ccc} C_\bullet(X \setminus B, A \setminus B) & \xrightarrow{\quad i_* \quad} & C_\bullet(X, A). \\ & \searrow \quad \nearrow & \\ & C_\bullet^{\mathscr{U}}(X, A) & \end{array}$$

and pass to homology. We obtain a commutative diagram

$$\begin{array}{ccc} H_n(X \setminus B, A \setminus B) & \xrightarrow{\quad i_* \quad} & H_n(X, A). \\ & \searrow \quad \nearrow_{\cong} & \\ & H_n^{\mathscr{U}}(X, A) & \end{array}$$

Observe that $C_\bullet(A \setminus B) = C_\bullet(A) \cap C_\bullet(X \setminus B)$. This means we can write

$$C_\bullet(X \setminus B, A \setminus B) = {C_\bullet(X \setminus B)}\big/{C_\bullet(A \setminus B)} = {C_\bullet(X \setminus B)}\big/{C_\bullet(A) \cap C_\bullet(X \setminus B)}$$

and it is immediate that

$$C_\bullet(X \setminus B, A \setminus B) \longrightarrow C_\bullet^{\mathscr{U}}(X, A) = {C_\bullet(X \setminus B) + C_\bullet(A)}\big/{C_\bullet(A)}$$

is an isomorphism. Combined, this implies that i_* is an isomorphism. □

We now return to question how to compute $H_n(X)$ form $H_n(A)$ and $H_n(X/A)$.

Definition 3.34. Let X be a space with a subset $A \subset X$.

(i) The subspace A is called a *retract* of X if there exists a continuous map $r\colon X \longrightarrow A$ such that $r|_A = \mathrm{id}_A$. Such a map r is called a *retraction.*

(ii) The subspace A is called a *deformation retract* of X if there exists a retraction $r\colon X \longrightarrow A$ such that $r \simeq \mathrm{id}_X$.

(iii) The subspace A is called a *strong deformation retract* of X if there exists a retraction $r\colon X \longrightarrow A$ such that $r \simeq_A \mathrm{id}_X$.

Definition 3.35. A pair (X, A) is called a *good pair* if A is closed and there exists an open subset $U \subset X$ such that $A \subset U$ and A is a strong deformation retract of U.

Proposition 3.36. *If (X, A) is a good pair then there exists a long exact sequence*

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{g_*} \widetilde{H}_n(X/A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \cdots$$

in reduced homology where $i\colon A \hookrightarrow X$ is the inclusion and $g\colon X \twoheadrightarrow X/A$ is the quotient map.

Proof. We have a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(A) \xrightarrow{i_*} \widetilde{H}_n(X) \xrightarrow{j_*} H_n(X, A) \longrightarrow \widetilde{H}_{n-1}(A) \longrightarrow \cdots$$

with the inclusion $j\colon (X, \emptyset) \hookrightarrow (X, A)$. Take an open neighbourhood U of A such that A is a strong deformation retract of U. For the triple (X, U, A) look at the short exact sequence

$$0 \longrightarrow C_\bullet(U, A) \longrightarrow C_\bullet(X, A) \longrightarrow C_\bullet(X, U) \longrightarrow 0$$

of chain complexes. We obtain a long exact sequence

$$\cdots \longrightarrow H_n(U, A) \longrightarrow H_n(X, A) \longrightarrow H_n(X, U) \longrightarrow H_{n-1}(U, A) \longrightarrow \cdots$$

But $H_\bullet(U, A) \cong H_\bullet(A, A) = 0$, hence the inclusion induces an isomorphism $H_n(X, A) \xrightarrow{\sim} H_n(X, U)$. Observe that the quotient map $q\colon X \longrightarrow X/A$ induces a homeomorphism

$$(X \setminus A, U \setminus A) \xrightarrow{\sim} ((X/A) \setminus (U/A), (U/A) \setminus (A/A))$$

and that there is a commutative diagram

$$\begin{array}{ccccc} H_n(X, A) & \xrightarrow{\sim} & H_n(X, U) & \xrightarrow{\sim} & H_n(X \setminus A, U \setminus A) \\ \downarrow q_* & & \downarrow q_* & & \cong \downarrow q_* \\ H_n(X/A, A/A) & \xrightarrow{\sim} & H_n(X/A, U/A) & \xrightarrow{\sim} & H_n((X/A) \setminus (A/A), (U/A) \setminus (A/A)). \end{array}$$

Hence, we obtain a commutative diagram

$$\begin{array}{ccc} \widetilde{H}_n(X) & \xrightarrow{j_*} & H_n(X, A) \\ & \searrow^{q_*} & \cong \downarrow q_* \\ & & \widetilde{H}_n(X/A) \end{array}$$

which gives the result. □

This may be applied to compute $H_\bullet(S^n)$. For $n = 0$ we have $H_0(S^0) = \mathbb{Z}^2$, $\widetilde{H}_0(S^0) = \mathbb{Z}$ and $H_n(S^0) = \widetilde{H}_n(S^0) = 0$ for $n \neq 0$. Inductively, (D^n, S^{n-1}) is a good pair and we obtain an exact sequence

$$\cdots \longrightarrow \widetilde{H}_k(D^n) \longrightarrow \widetilde{H}_k(D^n/S^{n-1}) \xrightarrow{\partial_k} \widetilde{H}_{k-1}(S^{n-1}) \longrightarrow \widetilde{H}_{n-1}(D^n) \longrightarrow \cdots$$

where $\widetilde{H}_k(D^n) = 0$ for all k. Hence, $\widetilde{H}_k(S^n) \cong \widetilde{H}_{k-1}(S^{n-1})$ which implies

$$\widetilde{H}_k(S^n) = \begin{cases} \mathbb{Z}, & \text{for } k = n \\ 0, & \text{otherwise.} \end{cases}$$

Corollary 3.37. *The sphere S^{n-1} is not a retract of D^n.*

Proof. Let $i\colon S^{n-1} \hookrightarrow D^n$ be the inclusion. Suppose $r\colon D^n \longrightarrow S^{n-1}$ were a retraction. Taking reduced homology we would get a commutative diagram

$$\begin{array}{ccc} \mathbb{Z} = \widetilde{H}_{n-1}(S^{n-1}) & \longrightarrow & \widetilde{H}_{n-1}(D^n) = 0 \\ & {\scriptstyle \mathrm{id}_{\mathbb{Z}}} \searrow & \downarrow \\ & & \widetilde{H}^{n-1}(D^{n-1}) = \mathbb{Z} \end{array}$$

which is impossible. □

3.5 Classical Theorems of Topology

Theorem 3.38 (Brouwer fixed–point theorem)**.** *Any continuous map $f\colon D^n \longrightarrow D^n$ has a fixed point.*

Proof. Suppose to the contrary that $f(x) \neq x$ for all $x \in D^n$. Consider the ray through x starting at $f(x)$. Let $r(x)$ be the unique point of intersection of this ray with $S^{n-1} = \partial D^n \subset D^n$. This clearly defines a retraction $r\colon D^n \longrightarrow S^{n-1}$. But this is impossible by Corollary 3.37. □

Theorem 3.39 (Invariance of Dimension)**.** *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be non–empty open sets. If U and V are homeomorphic as topological spaces then $n = m$.*

Proof. For $x_0 \in U$ excision implies that there is an isomorphism

$$H_k(U, U \smallsetminus \{x_0\}) \xrightarrow{\ \simeq\ } H_k(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{x_0\}).$$

There is a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_k(\mathbb{R}^n) \longrightarrow H_k(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{x_0\}) \longrightarrow \widetilde{H}_{k-1}(\mathbb{R}^n \smallsetminus \{x_0\}) \longrightarrow \widetilde{H}_{k-1}(\mathbb{R}^n) \longrightarrow \cdots$$

which implies that $H_k(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{x_0\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^n \smallsetminus \{x_0\}) \cong \widetilde{H}_{k-1}(S^{n-1})$ because $\mathbb{R}^n$ is contractible. In summary,

$$H_k(U, U \smallsetminus \{x_0\}) \cong \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n. \end{cases}$$

This implies the result. □

Let $f\colon S^n \longrightarrow S^n$ be continuous. We can look at the induced map $f_*\colon \widetilde{H}_n(S^n) \longrightarrow \widetilde{H}_n(S^n)$ on reduced homology. Because $\widetilde{H}_n(S^n) \cong \mathbb{Z}$ this is an endomorphism $\mathbb{Z} \longrightarrow \mathbb{Z}$ which is given by multiplication by some uniquely determined $n \in \mathbb{Z}$. We call $\deg(f) := n$ the *degree* of f. The degree satisfies a number of elementary properties:

(i) We always have $\deg(\mathrm{id}_{S^n}) = 1$.
(ii) The degree of f vanishes if f is not surjective, because for $x_0 \in S^n \smallsetminus f(S^n)$ the map f factors as $S^n \longrightarrow S^n \smallsetminus \{x_0\} \hookrightarrow S^n$. But $S^n \smallsetminus \{x_0\}$ is contractible which implies that $f_* = 0$ on reduced homology.
(iii) The degree is homotopy invariant.
(iv) The degree is multiplicative with respect to composition, i. e. $\deg(f \circ g) = \deg(f)\deg(g)$. In particular, any homotopy equivalence $f\colon S^n \longrightarrow S^n$ satisfies $\deg(f) = \pm 1$.

Lemma 3.40. *Let $F\colon \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ be the reflection at a hyperplane $H \subset \mathbb{R}^{n+1}$. In particular $F \in \mathrm{O}(n+1)$ with $\det(F) = -1$ and hence $F(S^n) \subset S^n$. Write $f = F|_{S^n}$. Then $\deg(f) = -1$.*

Proof. Without loss of generality assume $H = \{0\} \times \mathbb{R}^n$ and that F is given by the formula $F(x_1, \dots, x_{n+1}) = (-x_1, x_2, \dots, x_{n+1})$. In the case $n = 0$ write $S^0 = \{x\} \cup \{y\} \subset \mathbb{R}$ and observe that f just interchanges x and y. The reduced homology $\widetilde{H}_0(S^0) = \mathbb{Z}$ is generated by $[x - y]$. Hence, $f_*([x - y]) = [y - x] = -[x - y]$ which implies that $\deg(f) = -1$.

Inductively, consider the upper hemisphere $D^n_+ = \{x_{n+1} \geq 0\} \subset S^n$ whose boundary is $\partial D^n_+ = \{x_{n+1} = 0\} \cong S^{n-1}$. It is clear that $f(D^n_+) \subset D^n_+$ and $f(\partial D^n_+) = \partial D^n_+$. We have a good pair (D^n_+, S^{n-1}) and hence a long exact sequence

$$\cdots \longrightarrow \widetilde{H}_n(D^n_+) \longrightarrow \widetilde{H}_n(D^n_+/S^{n-1}) \longrightarrow \widetilde{H}_{n-1}(S^{n-1}) \longrightarrow \widetilde{H}_{n-1}(D^n_+) \longrightarrow \cdots$$

By naturality we have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{H}_n(D^n_+/S^{n-1}) & \xrightarrow{\sim} & \widetilde{H}_{n-1}(S^{n-1}) & \longrightarrow & 0 \\ & & \downarrow f_* & & \downarrow f_* & & \\ 0 & \longrightarrow & \widetilde{H}_n(D^n_+/S^{n-1}) & \xrightarrow{\sim} & \widetilde{H}_{n-1}(S^{n-1}) & \longrightarrow & 0 \end{array}$$

Now, by induction $f_*\colon \widetilde{H}_{n-1}(S^{n-1}) \longrightarrow \widetilde{H}_{n-1}(S^{n-1})$ is just multiplication by -1 and the commutativity of this diagram implies the result. □

Corollary 3.41.
(i) *For any orthogonal map $A \in \mathrm{O}(n)$ we have* $\deg(A|_{S^{n-1}}) = \det(A)$.
(ii) *The antipodal map $-\mathrm{id}\colon S^n \longrightarrow S^n$ has degree* $\deg(-\mathrm{id}) = (-1)^{n+1}$.
(iii) *If $f\colon S^n \longrightarrow S^n$ has no fixed point then* $\deg(f) = (-1)^{n+1}$.

Proof. We only prove (iii). If $f(x) \neq x$ for all $x \in S^n$ then

$$H(x,t) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}$$

defines a homotopy from f to the antipodal map $-\mathrm{id}$. □

Theorem 3.42 (Hairy Ball Theorem)**.** *The n–sphere S^n admits a continuous vector field without zeroes if and only if n is odd.*