Vorlesung aus dem Sommersemester 2013

Symplectic Geometry

Prof. Dr. Fabian Ziltener

TEXed by Viktor Kleen & Florian Stecker

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1 From Classical Mechanics to Symplectic Topology

1.1 Introducing Symplectic Forms

Definition 1.1. A *symplectic form* is a non–degenerate and closed 2–form on a manifold. *Non–degeneracy* means that if $\omega_x(v, w) = 0$ for all $w \in T_xM$, then $v = 0$. *Closed* means that $\mathrm{d}\omega = 0$.

Reminder. A differential k–form on a manifold M is a family of skew–symmetric k–linear maps

$$(\omega_x \colon T_xM^{\times k} \to \mathbb{R})_{x \in M},$$

which depends smoothly on x.

Definition 1.2. We define the *standard symplectic form* ω_0 on $\mathbb{R}^{2n}$ as follows: We identify the tangent space $T_x\mathbb{R}^{2n}$ with the vector space $\mathbb{R}^{2n}$. Given $v, w \in T_x\mathbb{R}^{2n} = \mathbb{R}^{2n}$, we denote by $v^1, v_2, v^3, v_4, \dots$ the standard coordinates of v. We define

$$\omega_{0,x}(v, w) = v^{2i-1}w_{2i} - v_{2i}w^{2i-1}$$

Remark 1.3. For $n = 1$ this is the usual area form on $\mathbb{R}^2$.

Exercise 1.4. We denote by $q^1, p_1, \dots, q^n, p_n \colon \mathbb{R}^{2n} \to \mathbb{R}$ the standard coordinate maps. Then the symplectic form ω_0 is given by

$$\omega_0 = \mathrm{d}q^i \wedge \mathrm{d}p_i$$

Remark 1.5. On S^{2n} there is no symplectic form if $n \geq 2$.

1.2 Hamiltonian mechanics

Consider a non–relativistic particle in Euclidean space $\mathbb{R}^{2n}$. We use the following notations:

q	position of the particle
t	time
$v \coloneqq \dot{q} = \frac{\mathrm{d}q}{\mathrm{d}t}$	velocity
$a \coloneqq \ddot{q} = \frac{\mathrm{d}^2q}{\mathrm{d}t^2} = \frac{\mathrm{d}v}{\mathrm{d}t}$	acceleration
m	mass
$p \coloneqq mv$	momentum of the particle
F	force exerted on the particle

Newton's second law states that $F = \dot{p}$. Consider the conservative case, i.e. there is a potential for the force; there exists a map $U \colon \mathbb{R}^n \to \mathbb{R}$ such that $F = -\nabla U$. We obtain the following equations of motion:

$$\dot{q} = \frac{p}{m} \tag{1}$$

$$\dot{p} = -\nabla U \tag{2}$$

These equations can again be rewritten by introducing the *Hamiltonian function*

$$H\colon \mathbb{R}^{2n} \to \mathbb{R}, \quad H(q^1, p_1, \dots, q^n, p_n) = \frac{\|p\|^2}{2m} + U(q).$$

The equations (1.1) and (1.2) are equivalent to *Hamilton's equations*:

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i} \end{aligned}$$

These equations in ture are equivalent to

$$\omega_0(\dot{x}, \_) = (\,\mathrm{d}H)_x$$

for the path $x = (q^1, p_1, \dots, q^n, p_n)\colon \mathbb{R} \to \mathbb{R}^{2n}$.

Remark 1.6. The space $\mathbb{R}^n$ of positions is called *configuration space* and $\mathbb{R}^{2n}$ is called *phase space*.

Example 1.7 (Stone / rigid body)**.** The state of a rigid body in $\mathbb{R}^3$ can be described by the configuration space $R^3 \times \mathrm{SO}(3)$. Its phase space is the cotangent bundle of the configuration space.

1.3 Overview over some topics of this course

Let X be a manifold. Then the cotangent bundle T^*X carries a canonical symplectic form ω_{can}.

In a classical mechanical system with symmetries, we can reduce the *degrees of freedom*, i.e. the dimension of the configuration space. Consider for example a system of 2 point particles in $\mathbb{R}^n$. We assume that the two particles interact by a conservative central force and that they have the same mass m. The potential then is of the form $V(\|q_1 - q_2\|)$. We can instead view this as a particle with position $q_1 - q_2$. This reduced system corresponds to a single point particle in $\mathbb{R}^n$ subject to a central force.

More generally, consider k point particles in $\mathbb{R}^n$, subject to conservative central forces between each two particles. The reduced system corresponds to a system of $k-1$ particles. The number of degrees of freedom is reduced from kn to $(k-1)n$. The motion of the original system can be described by the motion of the reduced system and the motion of the center of mass. The phase space of the reduced system is called the *symplectic quotient*. In general, this is a symplectic manifold which is constructed from a so called *Hamiltonian action* of a Lie group.

Example 1.8 (Lie groups)**.** $\mathrm{SO}(3)$, $\mathrm{SO}(n)$, $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{SU}(n)$, $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, $*$, $(\mathbb{R}^n, +)$, S^1, $(S^1)^k$, ...

Locally all symplectic manifolds look the same (Darboux' theorem), in contrast to Riemannian geometry (this is because $\mathrm{d}\omega = 0$, i.e. 'the symplectic curvature vanishes').

Important objects in symplectic geometry are *(co-)isotropic* submanifolds. In classical mechanics certain energy level sets are coisotropic submanifolds. Extreme cases of coisotropic submanifolds are *Lagrangian submanifolds.* These are submanifolds of half the dimension of a symplectic manifold on which the symplectic form vanishes. (for example 1–dimensional submanifolds of 2–manifolds). In mechanics configuration space is a Lagrangian submanifold of phase space.

Weinstein's theorem says that some neighbourhood of a given Lagrangian submanifold is symplectomorphic to a neighbourhood of the 0–section of the cotangent bundle of L.

1.4 Overview over some current questions

Nowadays, symplectic geometers mostly look at global properties of symplectic manifolds. A natural question is: How many symplectic forms are there on a given manifold (up to diffeomorphism)?

Moser's theorem implies that the total area of a given symplectic form on a closed surface is the only invariant.

One can ask if there is a symplectic form ω on $\mathbb{R}^{2n}$ such that for every smooth embedding $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ the pulled back form $\varphi^*\omega_0$ is not equal to ω. A more obvious questions would be: Is there a symplectic form on $\mathbb{R}^{2n}$ which is not diffeomorphic (symplectomorphic) to the standard form? Indeed, this is the case. Consider for example $\omega := \varphi^*\omega_0$ where $\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is some embedding with image B_1^{2n}. A form as in the first question is called *exotic*. Michael Gromov proved in 1985 that for $n \geq 2$ there is an exotic form on $\mathbb{R}^{2n}$. The proof involves a certain property of Lagrangian submanifolds of $\mathbb{R}^{2n}$ equipped with ω_0, which is established using *pseudoholomorphic curves.*

Let (M, ω) be a symplectic manifold. A *Hamiltonian diffeomorphism* is a map $\varphi\colon M \to M$ for which there is a smooth function $[0,1] \times M \to \mathbb{R}$ whose *Hamiltonian time–1–flow* equals φ.

More precisely, let $H\colon M \to \mathbb{R}$ be a smooth function. The Hamiltonian vector field H is defined to be the unique vector field X_H on M such that $\omega(X_H, \_) = \mathrm{d}H$. Let us consider a smooth function $H\colon [0,1] \times M \to \mathbb{R}$. We define the *Hamiltonian flow* of H to be the flow φ_H of the family of vector fields $(X_{H(t,\_)})_{t\in[0,1]}$.

Reminder. This flow is the unique smooth map

$$\varphi_H\colon [0,1] \times M \to M$$

that solves the equations

$$\varphi_H(0, \_) = \mathrm{id} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\varphi_H(t, x_0) = X_{H(t,\_)}(\varphi_H(t, x_0)).$$

The map $\varphi_H(1, \_)$ is called the *time–1–flow* of H.

Hamiltonian diffeomorphisms describe the time–1–evolution of a mechanical system if $M = T^*N$.

In the 1960s V.Arnold formulated the folliwing conjecture:

Conjecture. Every Hamiltonian diffeomorphism φ on a closed symplectic manifold has at least as many fixed points as a smooth function on M has at least critical points, i.e.

$$|\operatorname{Fix}(\varphi)| \geq \operatorname{Crit}(M) := \min\{|\operatorname{Crit}(f)| \colon f \in C^\infty(M)\}$$

Do we really need M to be closed? Yes, look at the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ with $H(q,p) = p/2$.

Exercise 1.9. $\operatorname{Crit}(M) \geq 2$ if M is closed and $\dim M > 0$.

Exercise 1.10. $\operatorname{Crit}(S^n) = 2$.

For $(S^2, \omega_{\mathrm{st}})$ the conjecture was proven in 1974. The conjecture led Andreas Floer to the definition of a homology whose generators are the fixed points of the Hamiltonian diffeomorphism.

Exercise 1.11. Let (M, ω) be a symplectic manifold and $H \colon \mathbb{R} \times M \to \mathbb{R}$ a smooth 1–periodic function. Then the fixed points of the time–1–flow of H correspond to the periodic orbits $x \colon \mathbb{R} \to M$ which satisfy Hamilton's equation $\omega(\dot{x}, \_) = \mathrm{d}H$. Periodic orbits are important in celestial mechanics.

Why did Arnold choose the lower bound $\operatorname{Crit}(M)$? If $x_0 \in M$ is a critical point of a function $F \colon M \to \mathbb{R}$, then $\varphi_F(t, x_0) = x_0$, so $X_F(x_0) = 0$.

Liouville's Theorem says that every symplectic embedding of an open subset of $\mathbb{R}^{2n}$ into $\mathbb{R}^n$ is volume preserving. Hence we are led to the following question: How much do symplectic embeddings and volume–preserving embeddings differ?

A famous result by Gromov shows that they differ a lot. It says that it is impossible to embed the open ball of radius $r > 1$ into the standard symplectic cylinder symplectically, although this is possible in a volume–preserving way.

Theorem 1.12 (Gromov)**.** *If $r > 1$ then there does not exist a symplectic embedding of B_r^{2n} into the standard symplectic cylinder $Z^{2n} = B_1^2 \times \mathbb{R}^{2n-2}$.*

Remark 1.13. There is an analogy to quantum mechancis: The statement of Gromov's theorem carries the spirit of Heisenberg's uncertainty principle: One cannot simultaneously measure both position and momentum of a particle. More precisely, the product of the standard deviations of position and momentum is bounded below by $\hbar/2$. Similarly, Gromov's non–squeezing theorem implies that we may not 'determine both q^i and p_i in a sharper way by changing coordinates in a canonical way.'

2 Linear (pre-)symplectic geometry

In this chapter we will investigate symplectic vector spaces and associated notions such as linear (pre-)symplectic maps and (co-)isotropic and Lagrangian subspaces.

These notions are linear analogues of notions associated with symplectic manifolds. We will encounter linear versions of some important results and constructions involving symplectic manifolds, e.g. Liouville's, Darboux' and Weinstein's theorems, the contangent bundle and symplectic reduction. The linear version of Liouville's theorem states that every linear symplectic map has determinant 1. The linear version of Darboux' theorem says that two symplectic vector spaces are isomorphic if and only if their dimensions are the same. The linear version of Weinstein's theorem says that, given two Lagrangian subsapces of a given symplectic vector space, there exists an automorphism of the symplectic vectorspace that carries one subspace to the other.

We will also see linear complex structures. Nonlinear versions of these structures play an important role in modern symplectic geometry, wher they are used to define pseudoholomorphic curves. Such 'curves' are used in the proof of Gromov's non–squeezing theorem. They build a bridge between symplectic and Riemannian geometry. In this chapter vector spaces will always be finite dimensional over $\mathbb{R}$.

2.1 (Pre-)symplectic vector spaces and linear (pre-)symplectic maps

Definition 2.1. We call a bilinear form $\omega\colon V \times V \to \mathbb{R}$ on a finite dimensional vector space V

- *skew–symmetric* if $\omega(v, w) = -\omega(w, v)$.
- *non–degenerate* if $\omega(v, w) = 0$ for all $w \in V$ implies $v = 0$.
- *symplectic* if it is skew–symmetric and non–degenerate.

If ω is skew–symmetric then (V, ω) is called a *presymplectic vector space.* If it is symplectic, (V, ω) is called a *symplectic vector space.* We define the map

$$\_^{\flat_\omega} : V \to V^* \qquad v \mapsto v^{\flat_\omega} := \omega(v, \_)$$

The *rank* and *corank* of ω are

$$\mathrm{rk}(\omega) = \dim \mathrm{im}\, \flat_\omega \qquad \mathrm{co\,rk}(\omega) = \dim \ker \flat_\omega.$$

Example 2.2. The zero form $\omega = 0$ is skew–symmetric and $\mathrm{rk}(\omega) = 0$ and $\mathrm{co\,rk}(\omega) = \dim V$.

Definition 2.3. The *standard linear symplectic form* ω_0 on $\mathbb{R}^{2n}$ is given by

$$\omega_0(v, w) = \sum_{i=1}^{n} (v^{2i-1} w_{2i} - v^{2i} w_{2i-1}).$$

Remark 2.4. The form ω_0 agrees with the form $\sum_{i=1}^n Q^i \wedge P_i$ where $Q^1, P_1, \ldots, Q^n, P_n$ are the canonical coordiante functions. The form ω_0 is non–degenerate, hence $\operatorname{rk}(\omega_0) = 2n$ and $\operatorname{co\,rk}(\omega_0) = 0$.

Definition 2.5. Let W be a vector space. The *canonical linear symplectic form on* $W \oplus W^*$ is defined by

$$\omega_W\left(\begin{pmatrix} w \\ \varphi \end{pmatrix}, \begin{pmatrix} w' \\ \varphi' \end{pmatrix}\right) = \varphi'(w) - \varphi(w').$$

Let V and V' be vector spaces, $k \geq 0$ and $\varphi\colon V^k \to \mathbb{R}$ and $\varphi'\colon (V')^k \to \mathbb{R}$ multilinear maps. Define the direct sum of φ and φ' to be the multilinear map

$$\varphi \oplus \varphi'\colon (V \times V')^k \to \mathbb{R}, \qquad (v_1, v_1', \ldots, v_k, v_k') \mapsto \varphi(v_1, \ldots, v_k) + \varphi'(v_1', \ldots, v_k').$$

Remark 2.6. For every pair $k, l \geq 0$ there exists a presymplectic vector space of rank $2k$ and corank l. An example is $(U \times U^* \times W, \omega_U \oplus 0)$.

Definition 2.7. Let (V, ω) and (V', ω') be presymplectic vector spaces. A linear map $\Phi\colon V \to V'$ is called *linear presymplectic* if it pulls back ω' to ω, i.e.

$$\Phi^*\omega' = \omega'(\Phi(\_), \Phi(\_)) = \omega.$$

Definition 2.8. If (V, ω) is presymplectic vector space then we call a linear presymplectic isomorphism $V \to V$ an *automorphism* of the presymplectic vector space (V, ω). The set of all automorphisms of (V, ω) is called $\operatorname{Aut}(V, \omega)$. We also write $\operatorname{Sp}(2n) \coloneqq \operatorname{Aut}(\mathbb{R}^{2n}, \omega_0)$. If (V, ω) and (V', ω') are symplectic, we call a linear presymplectic map $V \to V'$ simply *symplectic map*.

Exercise 2.9. If $f\colon (V, \omega) \to (V', \omega')$ is a linear symplectic map between symplectic vector spaces of the same dimension, then f is an isomorphism.

Exercise 2.10. A linear map $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is symplectic if and only if $\det \Phi = 1$.

Exercise 2.11. Define

$$J_0 = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}.$$

Then $\Phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is symplectic if and only if $\Phi^T J_0 \Phi = J_0$.

Proposition 2.12. $\operatorname{Aut}(V, \omega)$ *is a group.*

Symplectic vector spaces are classified by their dimension. Presymplectic vector spaces (V, ω) are classified by $\dim V$ and $\operatorname{rk} \omega$.

Theorem 2.13.

1. *Every symplectic vector space has even dimension.*

2. *Two symplectic vector spaces of the same dimension are isomorphic.*

Corollary 2.14.

1. *the rank of every presymplectic vector space is even.*
2. *Two presymplectic vector spaces of the same dimension and rank are isomorphic.*

Remark 2.15. In contrast to Corollary 2.14, a symmetric bilinear form can have any rank. Furthermore, if B and B' are symmetric bilinear forms of different signature on a vector sapce V, then (V, B) and (V, B') are not isomorphic.

Corollary 2.16. *For every n–dimensional presymplectic vector space (V, ω) there exists an integer $0 \leq k \leq n/2$ and a basis*

$$u_1, \ldots, u_k, v^1, \ldots, v^k, w_1, \ldots, w_{n-2k}$$

such that $\omega(u_i, v^j) = \delta_i^j$ and $\omega(a, b) = 0$ for every other (unordered) pair of basis vectors a, b.

Proof. Exercise 2 in Assignment 3.

For the proof of Theorem 2.13 we need the following:
Let (V, ω) be a presymplectic vector space with a subspace $W \subset V$.

Definition 2.17. We define the *ω–complement* W^ω of W as the linear subspace

$$W^\omega = \{v \in V : \forall w \in W : \omega(v, w) = 0\}.$$

If ω is non–degenerate then W^ω is called the *symplectic complement* of W.

Definition 2.18. The subspace W is called *symplectic* if the restriction of ω to W is non–degenerate.

Remark 2.19. The subspace W is symplectic if and only if $W \cap W^\omega = 0$.

Example 2.20. If $\omega = 0$ then $W^\omega = V$ and $(\mathbb{R} \times 0)^{\omega_0} = \mathbb{R} \times 0 \subset \mathbb{R}^2$.

Remark 2.21. The orthogonal complement $W^\perp$ of a linear subspace W of an inner product space is always transverse to W, i.e. $W \cap W^\perp = 0$.

Lemma 2.22. *For every symplectic vector space (V, ω) and every subspace $W \subset V$ we have*

$$\dim W + \dim W^\omega = \dim V.$$

Furthermore, if W is symplectic then the same holds for its symplectic complement.

Proof of Theorem 2.13.

- Let (V, ω) be a symplectic vector space other than $(0, 0)$. Then there exists a pair of vectors $u, v \in V$ such that $\omega(u, v) \neq 0$. The set $W = \mathbb{R}u + \mathbb{R}v$ is a 2–dimensional symplectic subspace of V. Hence, Lemma 2.22 implies that W^ω is a symplectic subspace of dimensin $\dim V - 2$. By induction, $\dim W^\omega$ is even, whence $\dim V \in 2\mathbb{Z}$.

- We call a basis $u_1, v^1, \dots, u_n, v^n$ of a symplectic vector space (V, ω) *canonical* or *symplectic basis* if

$$\omega(u_i, v^j) = \delta_i^j \qquad \omega(u_i, u_j) = \omega(v^i, v^j) = 0.$$

  It is enough to show that every symplectic vector space (V, ω) has a canonical basis. This is trivial for $\dim V = 0$ and easy for $\dim V = 2$. So assume $\dim V \geq 4$. Because ω is non–degenerate there are u, v such that $\omega(u, v) \neq 0$. Define $u_1 := \omega(u, v)^{-1} u$ and $v^1 := v$. the subspace $W = (u_1, v^1)$ of V is symplectic and of dimension $\dim W = 2$. Lemma 2.22 implies that W^ω is a linear symplectic subspace with $\dim W^\omega < \dim V$. By induction W and W^ω have canonical bases which together yield a canonical basis for $V = W \oplus W^\omega$. □

Lemma 2.22 is a consequence of the following:

Proposition 2.23. *Let (V, ω) be a presymplectic vector space and $W \subset V$ a lienar subspace. The following equality holds:*

$$\dim W + \dim W^\omega = \dim V + \dim(W \cap V^\omega)$$

Furthermore, if ω is non–degenerate and W is symplectic, W^ω is symplectic.

The proof of Proposition 2.23 uses the following:
Let V be a vector space and $W \subset V$ a subspace. We denote by

$$W^0 := \{\varphi \in V^* \colon \varphi(w) = 0 \, \forall w \in W\}$$

the annihilator of W (in V^*).

Lemma 2.24. *We have* $\dim W + \dim W^0 = \dim V$.

Proof. Assignment 3, Exercise 1.

Remark 2.25. Let V and W be vector spaces and $T \colon V \to W$ a linear map. Then we have $\ker T^* = (\operatorname{im} T)^0 \subset W^*$. This follows immediately from the definitions.

Proof of Proposition 2.23. We prove the first assertion. By Lemma 2.24 we have the formula $\dim W^\omega + \dim(W^\omega)^0 = \dim V$. Recall $\flat_\omega \colon V \to V^*$, $\flat_\omega(v) = \omega(v, \_)$. We denote by $\iota_W \colon W \to V$ the inclusion map, $T := \flat_\omega \iota_W \colon W \to V^*$, and by $\iota \colon V \to V^{**}$, $\iota(v)(\varphi) = \varphi(v)$ the canonical map.

We first claim that $\flat_\omega = -\flat_\omega^* \iota$. To prove this, let $v, v' \in V$. Since ω is skew–symmetric, we have

$$(\flat_\omega^* \iota)(v)(v') = \flat_\omega^*(\iota(v))(v') = \iota(v)(\flat_\omega v') = (\flat_\omega v')(v) = \omega(v', v) = -\omega(v, v') = -(\flat_\omega v)(v').$$

It follows from the claim that $\iota_W^* \flat_\omega = -\iota_W^* \flat_\omega^* \iota = -T^* \iota$ and therefore $W^\omega = \ker(\iota_W^* \flat_\omega) = \ker(T^* \iota)$. Since $\dim V < \infty$, the map ι is an isomorphism of vector spaces. Hence $W^\omega = \ker(T^* \iota)$ and the equality $\ker T^* = (\operatorname{im} T)^0$ imply that $\dim W^\omega = \dim \ker T^* =$

$\dim(\operatorname{im} T)^0$. The rank–nullity theorem for the map T states that $\dim W = \dim \operatorname{im} T + \dim \ker T$. Combining this and using Lemma 2.24 we obtain

$$\begin{aligned}\dim W + \dim W^\omega &= \dim \operatorname{im} T + \dim(\operatorname{im} T)^0 + \dim \ker T \\ &= \dim V^* + \dim \ker T \\ &= \dim V + \dim(W \cap V^\omega).\end{aligned}$$

This proves the first statement. To prove the second statement let W be a symplectic subspace of a symplectic vector space V. Then we have $V^\omega = 0$ and $W \cap W^\omega = 0$. Using the formula from the first statement it follows that $W + W^\omega = V$ $(*)$. Let $u \in W^\omega$ be such that $\omega(u, u') = 0$ for all $u' \in W^\omega$. We show that $u = 0$. Let $v' \in V$. By $(*)$ there exist vectors $w' \in W$ and $u' \in W^\omega$ such that $v' = w' + u'$. It follows that $\omega(u, v') = \omega(u, w') + \omega(u, u') = 0 + 0 = 0$. Since ω is non–degenerate it follows that $u = 0$. Hence $\omega|_{W^\omega}$ is non–degenerate and therefore W^ω is a symplectic subspace of V. ☐

For the proof of Corollary 2.14 (the classification of presymplectic vector spaces) the key ingredient is:

Lemma 2.26 (Splitting Lemma)**.** *Let (V, ω) be a presymplectic vector space and $W \subset V$ a symplectic subspace. Then the map*

$$W \times W^\omega \to V, \quad (w, w') \mapsto w + w'$$

is an isomorphism with respect to $(\omega|_W) \oplus (\omega|_{W^\omega})$ and ω.

Proof. The map $W \times W^\omega \to V$ is linear and by the definition of the (pre-)symplectic forms $\omega|_W$ and $\omega|_{W^\omega}$ and of W^ω, it pulls ω back to $\omega|_W \oplus \omega|_{W^\omega}$. Since W is symplectic, we have $W \cap W^\omega = 0$. It follows that the map is injective. To see that it is surjective, note that by Proposition 2.23 we have

$$\dim W + \dim W^\omega = \dim V + \dim(W \cap V^\omega).$$

Furthermore, we have $V^\omega \subset W^\omega$ and thus $W \cap V^\omega \subset W \cap W^\omega = 0$. so $\dim(W + W^\omega) = \dim V$ and the map $W \times W^\omega \to V$ is surjective by the rank–nullity theorem. ☐

Remark 2.27. This lemma says that we may split off a symplectic subspace from a presymplectic vector space. The statement is wrong if W is not sympletic. As an example consider $W = \mathbb{R} \times 0 \subset \mathbb{R}^2 = V$ equipped with the standard symplectic form ω_0.

Lemma 2.28. *The ranks of two isomorphic presymplectic vector spaces are the same.*

Proof. Assignment 3, Exercise 3.

Lemma 2.29. *The rank of the direct sum of two presymplectic vector spaces equals the sum of the ranks of the spaces.*

Proof. Assignment 3, Exercise 3.

Remark 2.30. the direct sum construction for multilinear forms is 'associative' in the following sense: If $\omega, \omega', \omega''$ are k–linear forms on vector space V, V', V'', then the forms $(\omega \oplus \omega') \oplus \omega''$ and $\omega \oplus (\omega' \oplus \omega'')$ agree under the canonical identification $(V \times V') \times V'' \cong V \times (V' \times V'')$. We write $V \times V' \times V''$ for either of these spaces and $\omega \oplus \omega' \oplus \omega''$ for the corresponding iterated direct sum.

Exercise 2.31. Prove that all two–dimensional symplectic vector spaces are isomorphic without using Theorem 2.13. (Assignment 3, Exercise 9)

Remark 2.32. Let (V, ω) be an n–dimensional presymplectic vector space and $k \in \{0, \ldots, \lfloor \frac{n}{2} \rfloor\}$ such that (V, ω) is isomorphic to $(\mathbb{R}^{2k} \times \mathbb{R}^{n-2k}, \omega_0 \oplus 0)$. Then $\operatorname{rk} \omega = 2k$. This is by Lemma 2.28 and Lemma 2.29 and the fact that

$$\operatorname{rk}(\mathbb{R}^{2k}, \omega_0) = \dim \operatorname{im} \flat_{\omega_0} = \dim \mathbb{R}^{2k} - \dim \ker \flat_{\omega_0} = 2k.$$

Proof of Corollary 2.14. Let (V, ω) be a presymplectic vector space of dimension n. We claim that then there exists an integer $k \in \{0, \ldots, \lfloor n/2 \rfloor\}$ such that (V, ω) is isomorphic to $(\mathbb{R}^{2k} \times \mathbb{R}^{n-2k}, \omega_0 \oplus 0)$. The above remark then proves the first statement of Corollary 2.14. We postpone the proof of the claim.

To prove the 'only if' part of the second statement, let (V, ω) and (V', ω') be isomorphic presymplectic vector spaces. It follows from the definition of an isomorphism that the dimensions of V and V' agree. Furthermore, by Lemma 2.28 the ranks of the spaces agree. To prove the 'if' part let (V, ω) be a presymplectic vector space. By the above claim and remark (V, ω) is isomorphic to $(\mathbb{R}^{2k} \times \mathbb{R}^{n-2k}, \omega_0 \oplus 0)$, where $2k := \operatorname{rk} \omega$. The 'if' part is a consequence of this and the part that the inverse of an isomorphism of presymplectic vector spaces is again an isomorphism and the same holds for the composition of two isomorphisms.

Now we prove the above claim. If $\omega = 0$ then the statement is true, hence assume $\omega \neq 0$. We choose vectors $u, v \in V$ such that $\omega(u, v) \neq 0$. Consider the subspaces $W_1 = \mathbb{R}u + \mathbb{R}v$ and $V_1 = W_1^{\omega}$ of V, which are equipped with the forms $\sigma_1 = \omega|_{W_1}$ and $\omega_1 = \omega|_{V_1}$. The subspace W_1 is symplectic. Therefore by the Splitting Lemma (Lemma 2.26), the pair (V, ω) is isomorphic to $(W_1 \times V_1, \sigma_1 \oplus \omega_1)$. Repeating this we obtain presymplectic vector spaces $(V_0, \omega_0) = (V, \omega)$, (W_1, σ_1), (V_1, ω_1), (W_2, σ_2), (V_2, ω_2), $\ldots$ such that $(V_i, \omega_i) \cong (W_{i+1} \times V_{i+1}, \sigma_{i+1} \oplus \omega_{i+1})$. Hence $V_0 \cong W_1 \times W_2 \times \cdots$. Since by hypothesis $\dim V < \infty$ the recursive construction has to stop, i.e. there exists $k \geq 1$ such that $\omega_k = \omega_{k-1}|_{V_k} = 0$. It follows that

$$(V, \omega) \cong (W_1 \times \cdots \times W_k \times V_k, \sigma_1 \oplus \cdots \oplus \sigma_k \oplus 0).$$

By the above exercise the (W_i, σ_i) are isomorphic to $(\mathbb{R}^2, \omega_0)$. □

Remark 2.33. There is a variant of the above proof which does not use Proposition 2.23 (the formula $\dim W + \dim W^{\omega} = \dim V + \dim(W \cap V^{\omega})$). This result was used in the proof of the Splitting Lemma (Lemma 2.26) which is a key ingredient of the poof of Corollary 2.14. More precisely, it was used to show that $W + W^{\omega} = V$ for every symplectic subspace $W \subset V$. If W is 2–dimensional, then we may show the formula by an easier argument, similar to Gram–Schmidt orthogonalisation.

Every symplectic vector space carries a natural linear volume form. This form is preserved unter linear symplectic maps. In the case of $\mathbb{R}^{2n}$ equipped with the standard symplectic form ω_0, this is the content of a linear version of Liouville's theorem.

Definition 2.34. Let (V, ω) be a presymplectic vector space of even dimension $2n$. We define the *canonical* $2n$*–linear form* to be

$$\Omega = \frac{1}{k!}\omega^{\wedge k}$$

Theorem 2.35. *This form* Ω *does not vanish if and only if* ω *is non–degenerate.*

Remark 2.36*.* A nonvanishing top–degree form on a vector space is called a *linear volume form.*

Remark 2.37*.* Every volume form Ω on a vector space V induces a orientation $\mathcal{O}$ on V by the formula $\mathcal{O} = [v_1, \ldots, v_n]$ where $v_1, \ldots, v_n$ is a basis of V for which $\Omega(v_1, \ldots, v_n)$ is positive. We will call two ordered bases B and B' equivalent if the linear transformation $V \to V$ which carries B to B' has positive determinant.

Theorem 2.35 has the following immediate consequence:

Corollary 2.38. *Let* (M, ω) *be a* $2n$*–dimensional symplectic manifold. Then the differential form* $\frac{1}{n!}\omega^{\wedge n}$ *does not vanish anywhere.*

Remark 2.39*.* This means that Ω is a volume form on M. Such a volume form determines an orientation on M.

Theorem 2.35 implies the following linear version of Liouville's theorem.

Corollary 2.40. *Every automorphism of a symplectic vector space has determinant 1.*

Proof. Let $\Phi \colon (V, \omega) \to (V, \omega)$ be an automorphism. Define $\Omega = \frac{1}{n!}\omega^{\wedge n}$. We have

$$\Phi^*\Omega = \frac{1}{n!}(\Phi^*\omega)^{\wedge n} = \frac{1}{n!}\omega^{\wedge n} = \Omega$$

Since $\Omega \neq 0$ it follows that $\det \Phi = 1$. □

For the proof of Theorem 2.35 we need

Lemma 2.41. *We have* $\frac{1}{n!}\omega_0^{\wedge n} = \Omega_0$*.*

Proof of Theorem 2.35. Theorem 2.13 implies that there is an isomorphism $\Phi \colon V \to \mathbb{R}^{2n}$ such that $\Phi^*\omega_0 = \omega$. It follows that $\Omega = \frac{1}{n!}(\Phi^*\omega_0)^{\wedge n} = \Phi^*\Omega_0$. Since $\Omega_0 \neq 0$ and Φ is an isomorphism, it follows that $\Omega \neq 0$. Assume on the other hand that ω is degenerate. Choose $0 \neq v \in V^\omega$ and vectors $v_2, \ldots, v_n$ such that $v, v_2, \ldots, v_{2n}$ is a basis of V. It follows that $\Omega(v, v_2, \ldots, v_{2n}) = 0$. □

Remark 2.42. By Assignment 2, Exercise 10, every $\Phi \in \mathrm{Sp}(2n)$ satisfies $J_0 = \Phi^T J_0 \Phi$ where

$$J_0 = \begin{pmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -1 \\ & & & 1 & 0 \end{pmatrix}.$$

It follows that $1 = \det J_0 = \det(\Phi^T J_0 \Phi) = \det(\Phi)\det(J_0)\det(\Phi)$, so $\det(\Phi) \in \mathbb{Z}^\times$. To exclude -1 requires much more work.

2.2 (Co-)isotropic and Lagrangian subspaces and the linear Weinstein Theorem

There is an important class of submanifolds of symplectic manifolds, namely the class of Lagrangia submanifolds. Such a submanifold is half–dimensional and the symplectic form vanishes on it. As an example the configuration space of a mechanical system naturally sits inside phase space as a Lagrangian submanifold. A more general class of submanifolds consists of the coisotropic submanifolds. Examples are (real) hypersurfaces. These occur in mechanics as level sets of Hamiltonian functions.

Let (V, ω) be a presymplectic vector space.

Definition 2.43. A subspace $W \subset V$ is called

- *isotropic* if $W \subset W^\omega$,
- *coisotropic* if $W^\omega \subset W$,
- *Lagrangian* if $W = W^\omega$.

Remark 2.44. W is isotropic if and only if $\omega|_W = 0$.

Example 2.45. Let U be a vector space. We denote by $\omega = \omega_U$ the canonical linear symplectic form on $U \times U^* = V$. Recall $\omega_U((u, \varphi), (u', \varphi')) = \varphi'(u) - \varphi(u')$. Let $U' \subset U$ be a linear subspace. Define $W = U' \times 0 \subset V$. Then W is an isotropic subspace: We have $W^\omega = U \times U'^0$ (where $U'^0 = \{\varphi \in U^* : \forall u \in U : \varphi(u) = 0\}$). Let now $W = U' \times U^* \subset V$. This is coisotropic: $W^\omega = 0 \times U'^0$. Lagrangian subspaces are $U \times 0$ and $0 \times U^*$.

Exercise 2.46. Let (V, ω) be a symplectic vector space with a hyperplane $W \subset V$. Then W is coisotropic.

Examples of Lagrangian subspaces can be produced using the following exercise:

Exercise 2.47. Let (V, ω) be a symplectic vector space and $\Phi \colon V \to V$ a linear map. Show that Φ is symplectic if and only if its graph is a Lagrangian subspace of $V \times V$ equipped with the symplectic form $(-\omega) \oplus \omega$.

Let (V, ω) be a presymplectic vector space with a linear subspace $W \subset V$.

Proposition 2.48.

- *If* W *is isotropic then* $\dim W \leq \frac{1}{2}(\dim V + \dim V^\omega)$.
- *If* W *is coisotropic then* $\dim W \geq \frac{1}{2}(\dim V + \dim V^\omega)$.
- *If* W *is Lagrangian then* $\dim W = \frac{1}{2}(\dim V + \dim V^\omega)$.

Proof. By Proposition 2.23 we have $\dim W + \dim W^\omega = \dim V + \dim(W \cap V^\omega)$. If $W \subset W^\omega$ then $2 \dim W \leq \dim W + \dim W^\omega$. If on the other hand $W^\omega \subset W$ then $2 \dim W \geq \dim W + \dim W^\omega$ and $V^\omega \subset W^\omega \subset W$, so $W \cap V^\omega = V^\omega$. □

Lagrangian subspaces are extreme examples of (co-)isotropic subspaces.

Proposition 2.49. *The following statements are equivalent:*

1. $W \subset V$ *is Lagrangian.*
2. W *is a maximal isotropic subspace.*
3. W *is a minimal coisotropic subspace.*

Exercise 2.50. Prove $1 \Leftrightarrow 3$.

Lemma 2.51. *Let* $W \subset V$ *be an isotropic subspace. Then* W *is contained in a Lagrangian subspace.*

Proof. If $W = W^\omega$, there is nothing to prove. Hence assume that $v \in W^\omega \setminus W$. Then $W_1 = W + \mathbb{R}v \subset V$ has dimension $\dim W + 1$ and is isotropic since $W \subset W^\omega$ and $v \in W^\omega$. Induction over codimension yields an index k for which $W_k = W_k^\omega$. □

Proof of Proposition 2.49. Let W be a Lagrangian and $U \supset W$ an isotropic subspace. Then

$$U^\omega \subset W^\omega = W \subset U \subset U^\omega.$$

Hence $W = U$. If on the other hand W is maximal isotropic, W is Lagrangian by Lemma 2.51. □

Remark 2.52. Lemma 2.51 implies that Lagrangian subspaces always exist. In fact there exist isotropic subspaces of dimensions $0, 1, \ldots, \frac{1}{2}(\dim V + \dim V^\omega)$.

The main result of this section is the following linear version of Weinstein's theorem It classifies Lagrangian subspaces of symplectic vector spaces.

Theorem 2.53 (Linear Weinstein)**.** *Let* (V, ω) *be a symplectic vector space with a Lagrangian subspace. Then there exists an isomorphism* $(W \times W^*, \omega_W) \to (V, \omega)$ *that carries* $W \times 0$ *to* W*.*

Remark 2.54. $(W \times W^*, \omega_W)$ is a linear version of the cotangent bundle. The subspace $W \times 0$ is a linear version of configuration space.

Remark 2.55. Let W and W' be vector spaces with an isomorphism $T\colon W \to W'$. Then the map

$$W \times W^* \to W' \times W'^*, \quad (w, \varphi) \mapsto (Tw, \varphi \circ T^{-1})$$

is an isomorphism with respect to the canonical symplectic structures. In physical terms a linear coordinate change in configuration space induces a canonical linear transformation on phase space.

Combining this remark with Theorem 2.53 and using the canonical isomorphism $(\mathbb{R}^{2n}, \omega_0) \to (\mathbb{R}^n \times (\mathbb{R}^n)^*, \omega_{\mathbb{R}^n})$ we obtain:

Corollary 2.56. *Let (V, ω) be a $2n$–dimensional symplectic vector space with a Lagrangian subspace $W \subset V$. The there exists an isomorphism*

$$\Phi\colon (\mathbb{R}^{2n}, \omega_0) \to (V, \omega)$$

satisfying $\Phi(\mathbb{R}^n \times 0) = W$.

As an application of Theorem 2.53 we may also classify general coisotropic subspaces of symplectic vector spaces.

Corollary 2.57. *Let (V, ω) be a symplectic vector space and $W \subset V$ and $W' \subset V$ be coisotropic subspaces. Then there exists an automorphism $\Phi\colon (V, \omega) \to (V, \omega)$ satisfying $\Phi(W) = W'$ if and only if $\dim W = \dim W'$.*

Proof. The 'only if' part is clear. Let $W \subset V$ be a coisotropic subspace. Choose a linear complement $U \subset W$ of W^ω: $U \oplus W^\omega = W$. Then U is symplectic. Therefore, by the Splitting Lemma, we have $(V, \omega) \cong (U \oplus U^\omega, \omega|_U \oplus \omega|_{U^\omega})$. Define $k := \dim W^\omega$ and $2m := \dim U$. Then $\dim W = 2m + k$ and using Lemma 2.22 we have $\dim U^\omega = \dim V - 2m = \dim V - \dim W + k = 2k$. It follows that W^ω is Lagrangian in U^ω, since

$$W^\omega = W^\omega \cap U^\omega \subset W \cap U^\omega \subset (W^\omega)^\omega \cap U^\omega = (W^\omega)^{\omega|_{U^\omega}},$$

so W^ω is maximal isotropic in U^ω (Lagrangian subspaces of symplectic vector spaces have half dimension). Hence by Corollary 2.56, there exists an isomorphism $\psi\colon (\mathbb{R}^{2k}, \omega_0) \to (U^\omega, \omega|_{U^\omega})$ satisfying $\psi(\mathbb{R}^k \times 0) = W^\omega$. On the other hand, by Theorem 2.13 there exists an isomorphism

$$(\mathbb{R}^{2m}, \omega_0) \to (U, \omega|U).$$

The cartesian product of this map and ψ is an isomorphism

$$\Phi\colon (\mathbb{R}^{2m+2k}, \omega_0 \oplus \omega_0) \to (U \times U^\omega, \omega|_U \oplus \omega|_{U^\omega}) \cong (V, \omega)$$

satisfying $\Phi(\mathbb{R}^{2m+k} \times 0) = W$. The statement follows. □

Lemma 2.58 (Lagrangian complement)**.** *Let (V, ω) be a symplectic vector space and $W \subset V$ a Lagrangian subspace. Then there exists a Lagrangian subspace $W' \subset V$ that is complementary to W, i.e. that satisfies $V = W \oplus W'$.*

Exercise 2.59 (Assignment 4, Exercise 5)**.** Let (V, ω) be a symplectic vector space, $W \subset V$ a Lagrangian subspace and $U \subset V$ a subspace that is complementary to W, i.e. satisfies $V = W \oplus U$. Then the map

$$W \to U^*, \quad w \mapsto \omega(w, \_)|_U$$

is an isomorphism of vector spaces.

The idea of the proof of Lemma 2.58 is to choose any complement $U \subset V$ of W and then correct it so that it becomes Lagrangian.

Proof of Lemma 2.58. We choose a subspace $U \subset V$ complementary to W. By the above exercise the map $T\colon W \to U^*$, $Tw := \omega(w, \_)|_U$ is an isomorphism. We also define $T'\colon U \to U^*$, $T'u := -\frac{1}{2}\omega(u, \_)|_U$. The subspace $W' := \{u + T^{-1}T'u \mid u \in U\}$ has dimension equal to $\dim W' = \dim U = \dim V - \dim W = \frac{1}{2}\dim V$.

We now prove that W' is isotropic. For every pair of vectors $u, u' \in U$, we have

$$\begin{aligned}\omega(u + T^{-1}T'u, u' + T^{-1}T'u') &= \omega(u, u') + \omega(u, T^{-1}T'u') + \omega(T^{-1}T'u, u') + 0\\ &= \omega(u, u') - (T'u')(u) + (T'u)(u')\\ &= \omega(u, u') - \frac{1}{2}\omega(u, u') - \frac{1}{2}\omega(u, u') = 0,\end{aligned}$$

so W' is isotropic and thus Lagrangian. Since U intersects W trivially, the same holds for W', so W' has the required properties. □

Remark 2.60. Alternatively, this lemma may be proven by using a linear complex structure J compatible with ω and setting $W' = JW$.

Proof of Theorem 2.53. We choose a Lagrangian subspace $W' \subset W$ as in Lemma 2.58. We have $\dim V = \dim W + \dim W'$. We define the map

$$\Phi\colon V \to W \times W^*, v \mapsto (w, \omega(\_, w')|_W),$$

where (w, w') is the unique pair satisfying $v = w + w'$. Since $V = W \oplus W'$, this map is well–defined. It is linear and satisfies $\Phi^*\omega_W = \omega$. It is injective, since $W \cap W' = 0$. Since $\dim V = \dim(W \times W^*)$, the map is also surjective. Hence it is an isomorphism of symlectic vector spaces. Furthermore, we have $\Phi(W) = W \times 0$. This proves the theorem. □

2.3 Linear symplectic reduction

Symplectci reduction corresponds to the reduction of degrees of freedom in classical mechanics. The symplectic quotient is the quotient of a certain coisotropic submanifold of a given symplectic manifold (corresponding to phase space) by the action of a Lie group.

A submanifold N of a symplectic manifold (M, ω) is called coisotropic if its tangent space at x is a coisotropic subspace of the tangent space to M and x, for every point

$x \in N$. The above group action corresponds to the symmetries of the mechanical system and the symplectic quotient corresponds to the reduced phase space.

In this section we consider a linear version of this quotient: The quotient of a given coisotropic subspace of a symplectic vector space by its symplectic complement. More generally, let (V, ω) be a presymplectic vector space. We denote the quotient space

$$\overline{V} := V_\omega := V/V^\omega$$

We define

$$\overline{\omega} := \omega^V : \overline{V} \times \overline{V} \to \mathbb{R}, \qquad \overline{\omega}(v + V^\omega, w + V^\omega) := \omega(v, w).$$

Exercise 2.61 (Assignment 4)**.** Prove that this map is well–defined, i.e. the right hand side does not depend on the choice of representatives v and w. Also show that it is a linear symplectic form.

Definition 2.62. The pair $(\overline{V}, \overline{\omega})$ is called the *(linear) symplectic quotient* of (V, ω).

Remark 2.63. This construction justifies the terminology 'presymplectic vector space' since it shows how to get a symplectic vector space out of such a space.

Example 2.64. Let (V, ω) be a symplectic vector space and V' a vector space. What is the symplectic quotient of $(V \times V', \omega \oplus 0)$? It is (V, ω).

Exercise 2.65. Find an isomorphism between (V, ω) and $(\overline{V \times V'}, \overline{\omega \oplus 0})$.

Remark 2.66. Let (V, ω) be a presymplectic vector space and $W \subset V$ a linear subspace. The restriction of ω to W is again a presymplectic vector space. Hence the symplectic quotient of $(W, \omega|_W)$ is well–defined. It is given by $(W/(W \cap W^\omega), \overline{\omega|_W})$, since $W^{\omega|_W} = W \cap W^\omega$. Assume that ω is in fact coisotropic. Then $W^{\omega|_W} = W^\omega$. Hence in this case the quotient above equals $(W/W^\omega, \overline{\omega|_W})$. Its dimension equals

$$\dim W - \dim W^\omega = 2 \dim W - \dim V = \dim V - \operatorname{codim} W.$$

Example 2.67. The symplectic quotient of a hyperplane W in (V, ω) has dimension $\dim V - 2$.

2.4 Linear complex structures and the symplectic linear group

Let V be a (finite dimensional real) vector space.

Definition 2.68. A *(linear) complex structure* on V is an endomorphism of V, i.e. a linear map $J \colon V \to V$ such that $J^2 = -\operatorname{id}_V$.

Remark 2.69. If J is a complex structure on V, then V is a complex vector space via the scalar product

$$\mathbb{C} \times V \to V, \quad (a + ib, v) \mapsto (a + ib)v := av + bJv.$$

Hence the real dimension of V is twice the complex dimension of V. This is an even integer.

Complex structures are classified as follows:

Proposition 2.70 (Classification of linear complex structures)**.** *Let V be a vector space. Then for any pair of complex structures J and J' on V there exists an automorphism Φ of V satisfying $J' = \Phi^{-1} J \Phi$.*

Proof. We define complex scalar multiplications

$$\begin{aligned} m &\colon \mathbb{C} \times V \to V, & m(a+ib, v) &:= a + bJv \\ m' &\colon \mathbb{C} \times V \to V, & m'(a+ib, v) &:= a + bJ'v \end{aligned}$$

Denoting by $\cdot\colon \mathbb{R} \times V \to V$ the real scalar product of V. We have

$$2 \dim_{\mathbb{C}}(V, +, m) = \dim_{\mathbb{R}}(V, +, \cdot) = 2 \dim_{\mathbb{C}}(V, +, m').$$

Therefore, there exists an isomorphism of complex vector spaces

$$\Phi\colon (V, +, m) \to (V, +, m')$$

It satisfies $\Phi J v = \Phi m(i, v) = m'(i, \Phi v) = J' \Phi v$ for all $v \in V$. This implies $J' = \Phi^{-1} J \Phi$. □

In the following definition ω denotes a linear symplectic form, J a linear complex structure and g an inner product. (By definition, g is positive definite).

Definition 2.71.

1. We call the pair (ω, J) *compatible* if the bilinear form $\omega(\_, J\_)$ is an inner product.
2. We call the pair (J, g) *compatible* if the bilinear form $-g(\_, J\_)$ is symplectic.
3. We call the pair (g, ω) *compatible* if $-\flat_{\omega}^{-1} \flat_g \colon V \to V$ is a complex structure.
4. We call the triple (ω, J, g) *compatible* if $g = \omega(\_, J\_)$.

Remark 2.72. A short form fo the above definition is the following: Consider a pair consisting of two objcts of the following types:

$$\{\text{symplectic form, complex structure, inner product}\}$$

such that the two objects have different type. Such a pair is called compatible if and only if the third object defined by the condition $g = \omega(\_, J\_)$ is an object of the third type.

Example 2.73. The standard triple on $V = \mathbb{R}^{2n}$ is given by (ω_0, J_0, g_0) where g_0 denotes the standard inner product on $\mathbb{R}^{2n}$. This is a compatible triple.

Exercise 2.74. Let V be a real vector space, ω a symplectic form on V and J a complex structure on V. The following are equivalent:

1. The bilinear form $g_J := \omega(\_, J\_)$ on V is symmetric.

2. The form ω is invariant under J, i.e. $J^*\omega = \omega$.

3. The form g_J is invariant under J, i.e. $J^*g_J = g_J$.

It follows that the pair (ω, J) is compatible if and only if ω is invariant under J and the inequality $\omega(v, Jv) > 0$ holds for every $v \neq 0$.

Exercise 2.75. Compatible complex structures exist on every symplectic vector space.

Theorem 2.76. *Let V be a vector space and (ω, J, g) and (ω', J', g') compatible triples on V. Then there exists an automorphism $\Phi\colon V \to V$ which intertwines the two triples, i.e. it satisfies $\Phi^*\omega' = \omega$, $\Phi^*J' = J$ and $\Phi^*g' = g$.*

The proof of this result is based on the following:

Lemma 2.77. *Let V be a vector space, and ω and J symplectic and complex structures on V. Then (ω, J) is compatible if and only if the map*

$$h\colon V \times V \to \mathbb{C}, \qquad (v, w) \mapsto \omega(v, Jw) + i\omega(v, w)$$

is a Hermitian inner product on V with respect to complex scalar multiplication

$$\mathbb{C} \times V \to V, \qquad (a + ib, v) \mapsto (a + bJ)v.$$

Proof. Let (ω, J, g) be a compatible triple on V. We denote by m the complex scalar multiplication as above ad define h as above. We denote by $2n$ the real dimension of V. By Lemma 2.77, h is a Hermitian inner product on V. Hence by Gram–Schmidt orthonormalisation there exists a basis $v_1, \dots, v_n$ of V that is unitary with respect to h. Define

$$\Phi\colon \mathbb{R}^{2n} \to V, (q^1, p_1, \dots, q^n, p_n) \mapsto \sum_{j=1}^{n} (q^j v_j + p_j J v_j).$$

The identities $h(v_j, v_k) = \delta_{jk}$ and $h(v_j, Jv_k) = ih(v_j, v_k)$ imply that $\Phi^*\omega = \omega_0$ and $\Phi^*J = J_0$. Hence the claim follows. □

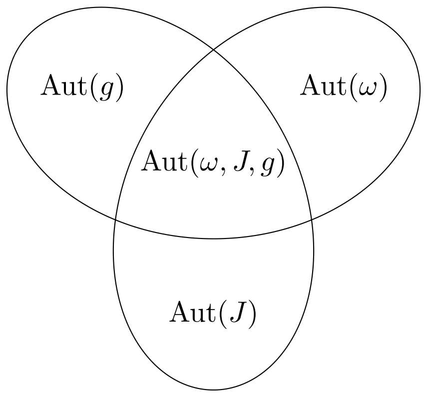
Compatible complex structures help understand the group of automorphisms of a symplectic vector space.

Definition 2.78. Let V be a vector space and (ω, J, g) a compatible triple on V. We denote by $\mathrm{Aut}(V, A_1, \dots, A_k) = \mathrm{Aut}(A_1, \dots, A_k)$ teh set of all vector space automorphisms preserving $A_1, \dots, A_k$. In particular

- The *linear symplectic automorphisms* $\mathrm{Aut}(\omega) = \mathrm{Sp}(V, \omega)$
- The *general linear group* $\mathrm{Aut}(J) = \mathrm{GL}_{\mathbb{C}}(V)$ of the complex space V.
- The *orthogonal group* $\mathrm{Aut}(g) = \mathrm{O}(V, g)$ of (V, g).
- The *unitary group* $\mathrm{Aut}(\omega, J, g)$ of (V, ω, J, g).

Proposition 2.79 (Trefoil decomposition)**.** *Let V be a vector space and (ω, J, g) a compatible triple. Then the following identites hold:*

$$\mathrm{Aut}(\omega, J, g) = \mathrm{Aut}(\omega, J) = \mathrm{Aut}(J, g) = \mathrm{Aut}(g, \omega)$$



Remark 2.80. In this result the structures ω, J, g play symmetric roles. However, as we say earlier, they differ a lot.

Proof of Proposition 2.79. It suffices to prove that $\mathrm{Aut}(\omega, J) \subset \mathrm{Aut}(g)$ and the analogous two other includsions hold. Observe that for $\Phi \in \mathrm{Aut}(\omega, J)$ we have

$$\Phi^* g = \omega(\Phi\_, J\Phi\_) = \omega(\Phi\_, \Phi J\_) = \omega(\_, K\_) = g,$$

i.e. $\Phi \in \mathrm{Aut}(g)$. ☐

Proposition 2.79 has the following application: We denote by $\mathrm{U}(n)$ the unitary group in (complex) dimension n, by $\mathrm{Sp}(2n)$ the group of symplectic $2n \times 2n$–matrices and by $\mathrm{GL}(n, \mathbb{C})$ the complex general linear group in dimension n, by $O(2n)$ the group of orthogonal $2n \times 2n$–matrices. We define

$$\Phi \colon \mathbb{C}^{n\times n} \to \mathbb{R}^{2n\times 2n}, \qquad A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

Corollary 2.81. *The equalities*

$$\Phi(\mathrm{U}(n)) = \mathrm{Sp}(2n) \cap \Phi(\mathrm{GL}(n, \mathbb{C})) = \Phi(\mathrm{GL}(n, \mathbb{C})) \cap \mathrm{O}(2n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n)$$

hold.

Here in the definition of $\mathrm{Sp}(2n)$ we reorder the canonical coordinates as $Q^1, \dots, Q^n$, $P_1, \dots, P_n$. This enables us to define Φ by the above formula simplifying notation. Note that $\mathrm{Sp}(2n) \cap \Phi(\mathrm{GL}(n, \mathbb{C})) = \mathrm{Sp}(2n) \cap \Phi(\mathbb{C}^{n\times n})$ and $\Phi(\mathrm{GL}(n, \mathbb{C})) \cap \mathrm{O}(2n) = \Phi(\mathbb{C}^{n\times n}) \cap \mathrm{O}(2n)$.

Remark 2.82. Identifying a matrix $M' \in \mathbb{C}^{n\times n}$ with its image under Φ, we may rewrite the above equality as

$$\mathrm{U}(n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n).$$

Note that Φ is an injective real linear map, hence this identification makes sense.

Definition 2.83. A *Lie group* is a group G with the structure of a smooth manifold such that inversion and multiplication are smooth maps.

Example 2.84 (Assignment 5)**.**

- $(\mathbb{R}, +)$ together with the standard manifold structure on $\mathbb{R}$.
- The circle $S^1 \subset \mathbb{C}$ together with the complex multiplication and the structure of a submanifold of $\mathbb{C} = \mathbb{R}^2$.
- The general linear group $\mathrm{GL}(V)$ of a real vector space V. $\mathrm{GL}(V)$ is open in $\mathrm{End}(V)$, hence it is a submanifold.

Remark 2.85. If G and G' are Lie groups then the product group $G\times G'$ with the product manifold structure is again a Lie group. Let $n \in \mathbb{N}$. It follows that $(\mathbb{R}^n, +)$ with the standard manifold structure on $\mathbb{R}^n$ is a Lie group.

Furthermore, the torus $(S^1)^n$ together with the product manifold structure is a Lie group.

Example 2.86. Consider $G = \mathbb{R}$ together with the standard manifold structure, the identity 0 and the composition $x \circ y = \sqrt[3]{x^3 + y^3}$. The map $x \mapsto 1 \circ x$ is not smooth, hence $\circ$ is not smooth. So $(\mathbb{R}, \circ, \text{standard manifold structure})$ is not a Lie group.

Theorem 2.87 (Cartan)**.** *Let G be a Lie group and $H \subset G$ a subgroup that is closed as a subset. Then H is an embedded submanifold of G.*

Proof. See Theorem 3.6, p. 47, Lie groups and Lie Algebras I, Encyclopaedia of Math. Sciences, Vol. 20, Springer 1993.

Definition 2.88. We call a subgroup of a Lie group that is an embedded submanifold an *embedded Lie subgroup.*

Remark 2.89. Let $H \subset G$ be an embedded Lie subgroup. eThen H together with the submanifold structure is a Lie group. This follows from the fact that the restrictions of the composition and inverse maps to H are smooth.

Example 2.90. Let V be a real vector space. It follows from Theorem 2.87 that every closed subgroup of $\mathrm{GL}(V)$ is an embedded Lie subgroup. In particular, let V be a real vector space and (ω, J, g) a compatible triple on V. Then for $A \in \{\omega, J, g\}$ the group $\mathrm{Aut}(V, A)$ is a closed subset of $\mathrm{GL}(V)$ (Check this!). Hence it is a submanifold of $\mathrm{GL}(V)$ and a Lie group, when endowed with the submanifold structure. Similarly, $\mathrm{Aut}(\omega, J, g)$ is a Lie group.

Let (ω, J, g) be a compatible triple on a vector space V.

Theorem 2.91. *The following holds:*

- $\mathrm{Aut}(\omega, J, g)$ *is a smooth strong deformation retract of* $\mathrm{Aut}(\omega)$*, i.e. there exists a smooth map* $h\colon [0,1] \times \mathrm{Aut}(\omega) \to \mathrm{Aut}(\omega)$ *such that*

$$h_0 = \mathrm{id}_{\mathrm{Aut}(\omega)}, \quad h_t|_{\mathrm{Aut}(\omega,J,g)} = \mathrm{id}_{\mathrm{Aut}(\omega,J,g)}, \quad \mathrm{im}\, h_1 \subset \mathrm{Aut}(\omega, J, g)$$

  for all $t \in [0,1]$*, where* $h_s(x) \coloneqq h(s,x)$*.*

- *Every compact subgroup of* $\mathrm{Aut}(\omega)$ *is contained in a subgroup which is conjugate to* $\mathrm{Aut}(\omega, J, g)$*.*

Remark 2.92. Theorem 2.91 says that the 'main' topological information about $\mathrm{Aut}(\omega)$ is contained in the subgroup $\mathrm{Aut}(\omega, J, g)$. The second part of the theorem implies that $\mathrm{Aut}(\omega, J, g)$ is a maximal compact subgroup of $\mathrm{Aut}(\omega)$, i.e. it is not contained in a bigger compact subgroup.

Example 2.93. In the symplectic vector space $(\mathbb{R}^2, \omega_0)$, we have $\mathrm{Aut}(\omega_0) = \mathrm{SL}(2,\mathbb{R}) \supset \mathrm{Aut}(\omega_0, J_0, g_0) = S^1$.

By a g–positive linear map we mean a linear map $\Phi\colon V \to V$ that is self–adjoint and positive definite with respect to g. This means that $g(v, \Phi v) > 0$ for all $v \in V \setminus 0$. We denote $n \coloneqq \dim V$. Let $\Phi\colon V \to V$ be a g–positive map. For $a \in \mathbb{R}$ we define the a^{th} power of Φ to be the linear map

$$\Phi^a \coloneqq T \begin{pmatrix} \lambda_1^a & & \\ & \ddots & \\ & & \lambda_n^a \end{pmatrix} T^{-1}$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of Φ (with multiplicities) and $T\colon \mathbb{R}^n \to V$ is any isomorphism such that $\Phi = T\,\mathrm{diag}(\lambda_1, \dots, \lambda_n)T^{-1}$.

We may define $Te_i \coloneqq v_i$ where $v_1, \dots, v_n$ are the eigenvectors. To see that the map Φ^a is well–defined, observe that $\lambda_i > 0$, since Φ is positive, hence λ_i^a makes sense. Furthermore, the definition does not depend on the choice of T (Check this!). The map Φ^a itself is g–positive. Furthermore, if $\Phi \in \mathrm{Aut}(V)$, we denote by $\Phi^{*g} = \Phi^*$ the g–adjoint map of Φ ($g(v, \Phi w) = g(\Phi^{*g} v, w)$ for all $v, w \in V$).

Exercise 2.94. Let $\Phi \in \mathrm{Aut}(V)$. Then the map $\Phi\Phi^*$ is g–positive and the map $(\Phi\Phi^*)^{-1/2}\Phi$ is orthogonal.

Let ω be a symplectic structure on V and g a compatible inner product.

Lemma 2.95. *The g–adjoint of a symplectic automorphism of V is a symplectic automorphism of V.*

Proof. Assignment 5, Exercise 14.

Lemma 2.96. *Let Φ be a linear symplectic g–positive automorphism on V. Then for every number $a \in \mathbb{R}$ the linear transformation $\Phi^a\colon V \to V$ is symplectic.*

For the proof of Lemma 2.96 we need the following: Let (V, ω) be a symplectic vector space. We write $V^{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ and $\omega^{\mathbb{C}}$ for the complex bilinear form on $V^{\mathbb{C}}$ induced by ω.

Lemma 2.97. *Let $\Phi \in \mathrm{Aut}(V, W)$, $\lambda, \lambda' \in \mathbb{C}$ and $v, v' \in V^{\mathbb{C}}$ such that $\Phi v = \lambda v$, $\Phi v' = \lambda' v'$ and $\lambda\lambda' \neq 1$. Then $\omega^{\mathbb{C}}(v, v') = 0$.*

Proof of Lemma 2.96. Let $\lambda, \lambda' \in (0, \infty)$ and $v, v' \in V$ be such that $\Phi v = \lambda v$, $\Phi v' = \lambda' v'$. Then $\omega(\Phi^a v, \Phi^a v') = (\lambda\lambda')^a \omega(v, v') = \omega(v, v')$. In the case $\lambda\lambda' \neq 1$ the last equality follows from Lemma 2.97. Since Φ is g–self–adjoint, the space V is the direct sum of the eigenspaces of Φ. Therefore this implies that Φ^a is symplectic. □

Proof of Lemma 2.97. $\lambda\lambda'\omega^{\mathbb{C}}(v, v') = \omega^{\mathbb{C}}(\Phi v, \Phi v') = \omega^{\mathbb{C}}(v, v')$. □

Proposition 2.98. *Let V be a vector space and G a compact subgroup of $\mathrm{Aut}(V)$. Then there exists a G–invariant inner product on V, i.e. an inner product g such that $\Phi^* g = g$ for all $\Phi \in G$.*

For a vector space V we denote by $\mathrm{Met}(V)$ the set of all inner products on V. Let (V, ω) be a symplectic vector space. We denote $J(V, \omega)$ the ω–compatible complex structures on V.

Proposition 2.99. *There exists a continuous map $r\colon \mathrm{Met}(V) \to J(V, \omega)$ such that*

$$r(\omega(\_, J\_)) = J, \qquad r(\Phi^* g) = \Phi^* r(g) \qquad \forall J \in J(V, \omega), g \in \mathrm{Met}(V), \Phi \in \mathrm{Aut}(V, \omega).$$

Proof of Theorem 2.91. We define the map

$$h\colon [0,1] \times \mathrm{Aut}(\omega) \to \mathrm{Aut}(\omega), \qquad \Phi \mapsto (\Phi\Phi^*)^{-t/2}\Phi.$$

We show that h is well–defined. Let $t \in [0,1]$ and $\Phi \in \mathrm{Aut}(\omega)$. By Exercise 2.94 the map $\Phi\Phi^*$ is positive and hence $(\Phi\Phi^*)^{-t/2}$ makes sense. By Lemma 2.95 and Lemma 2.96 this map and hence $h(t, \Phi)$ is symplectic. Hence h is well–defined. The condition $h_0 = \mathrm{id}_{\mathrm{Aut}(\omega)}$ is clearly satisfied. By Exercise 2.94 we have $h(1, \Phi) \in \mathrm{Aut}(g)$, i.e. $h_1(\mathrm{Aut}(\omega)) \subset \mathrm{Aut}(\omega, J, g)$ by the Trefoil Proposition. The condition $h(t, \Phi) = \Phi$ for $t \in [0,1]$ and $\Phi \in \mathrm{Aut}(\omega, J, g)$ follows from the fact $\Phi\Phi^* = \mathrm{id}_V$ for $\Phi \in \mathrm{Aut}(g) = \mathrm{O}(V, g)$. This proves the first part of Theorem 2.91.

Now let $G \subset \mathrm{Aut}(V, \omega)$ be a compact subgroup. By Proposition 2.98 there exists a G–invariant inner product g' on V. We choose a map r as in Proposition 2.99 and define $J' = r(g') \in J(V, \omega)$. By Theorem 2.76 there exists an automorphism $\psi \in \mathrm{Aut}(V, \omega)$ satisfying $\psi^* J = J'$. Since g' is G–invariant and r satisfies $r(\Phi^* g) = \Phi^* r(g)$ we have $\Phi^* J' = r(\Phi^* g') = J'$ for every $\Phi \in G$. Hence $G \subset \mathrm{Aut}(V, \omega, J')$. It follows that $\psi G \psi^{-1} \subset \mathrm{Aut}(V, \omega, J)$. □

Let V be a 2–dimensional vector space. We denote by $J(V)$ the set of complex structures on V, by $\Omega(V)$ the set of linear symplectic forms on V and for $\omega \in \Omega(V)$ by $J(V,\omega)$ the complex structures on V compatible with ω.

We choose a vector $v \in V \smallsetminus 0$. Then the map

$$J\colon V \smallsetminus \mathbb{R}v \to J(V), \qquad J(w)(av+bw) := aw - bv$$

is a bijection. We now fix a symplectic form ω on V. Then the symplectic forms on V correspond to $\mathbb{R}^\times$ via the bijection

$$\mathbb{R}^\times \to \Omega(V), \qquad a \mapsto a\omega.$$

The ω–compatible linear complex structures correspond to the vectors in the 'positive half–plane': We fix $v \in V \smallsetminus 0$ and denote by $H \subset V$ the open half–plane with boundary $\mathbb{R}v$ such that $\omega(v,w) > 0$ for all $w \in H$. The map J as above restricts to a bijection between H and $J(V,\omega)$. Hence in two dimensions compatibility only means that ω and J induce the same orientation of V.

Exercise 2.100. Characterise compatibility of (J,g) and (g,ω) in two dimensions.

Theorem 2.101. *For every compact Lie gorup G there exists a unique left–invariant measure $\mu = \mu_G$ on the Borel σ–algebra of G which satisfies $\mu(G) = 1$.*

Proof of Proposition 2.98. Since G is a closed subgroup of $\mathrm{Aut}(V)$, by Theorem 2.87 G is an embedded Lie subgroup. Hence it is a compact Lie group. Denote by μ the Haar measure on G. WE choose an inner product G_0 on V and define

$$g := \int_{\Phi \in G} \Phi^* g_0 \, \mathrm{d}\mu$$

This is a G–invariant inner product. □

Proof of Proposition 2.99. We define the map $r\colon \mathrm{Met}(V) \to J(V,\omega)$ as follows. Let $g \in \mathrm{Met}(V)$. We define $\Phi\colon V \to V$ to be the unique map satisfying $\omega = g(\Phi\_,\_)$. Since g is non–degenerate, this map is well–defined. We denote by Φ^* the g–adjoint of Φ. Since ω is skew–symmetric, the map Φ is g–skew–adjoint, i.e. $\Phi^* = -\Phi$. It follows that $P = \Phi^*\Phi = -\Phi^2$. This map is g–positive. We define $r(g) := \Phi^{-1/2}\Phi$.

We show that this map lies in $J(V,\omega)$. To see that it is a complex structure, we write $2n = \dim V$. Since $i\Phi$ is self–adjoint with respect to the Hermitian inner product on $V^{\mathbb{C}}$ induced by g, there exists an (orthonormal) complex linear transformation $T\colon \mathbb{C}^{2n} \to V^{\mathbb{C}}$ such that $T^{-1}i\Phi T$ is diagonal. Since $P = -\Phi^2$, it follows that Φ commutes with $P^{-1/2}$ and therefore $r(g)^2 = P^{-1}\Phi^2 = \mathrm{id}$.

To see that $r(g)$ is ω–compatible note that

$$\omega(\_, r(g)\_) = g(\Phi\_, P^{-1/2}\Phi\_) = \psi^* g$$

where $psi = P^{-1/2}\Phi$. Since ψ is invertible, it follows that $\omega(\_, r(g)\_)$ is an inner product on V.

Hence the map $r\colon \mathrm{Met}(V) \to J(V,\omega)$ is well–defined. The proof of continuity can be found in MacDuff–Sullivan, Exercise 2.52, p. 67. The properties $r(\omega(\_, J\_)) = J$ and $r(\psi^* g) = \psi^* r(g)$ follow from the defining equation of $r(g)$. □

Remark 2.102. The Haar measure always exists, but sometimes it is not right–invariant. For Lie groups this happens precisely if there exists $g \in G$ such that $\det \mathrm{Ad}_g \neq 1$.

Example 2.103. Consider the Lie group

$$\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\} \subset \mathrm{GL}_2(\mathbb{R}).$$

The left Haar measure is not right–invariant.

Let V be a real vector space, $\Phi \colon V \to V$ a linear map and $\lambda \in \mathbb{C}$. We define $m(\lambda) = m(\lambda, \Phi)$ to be the algebraic multiplicity of λ as an eigenvalue of Φ. Note that λ is an eigenvanlue of Φ if and only if $m(\lambda) > 0$.

Proposition 2.104. *Let (V, ω) be a symplectic vector space, $\Phi \in \mathrm{Aut}(V, \omega)$ and $\lambda \in \mathbb{C}$. Then*

$$m(\lambda) = m(\overline{\lambda}) = m(\lambda^{-1}) = m(\overline{\lambda^{-1}}).$$

Furthermore $m(\pm 1)$ is even.

Remark 2.105. In the case $|\lambda| = 1$ we have $\lambda^{-1} = \overline{\lambda}$.

Remark 2.106. Not every symplectic matrix is diagonalisable over $\mathbb{C}$. Take a non–diagonalisable automorphism T of a vector space W and define $\Phi = (T, T^*) \colon W \times W^* \to W \times W^*$. This gives examples of non–diagonalisable symplectic transformations with eigenvalues on $S^1 \setminus \{\pm 1\}$.

Proof of Proposition 2.104. Since Φ is a real endomorphism of a real vector space V, we have $m(\lambda) = m(\overline{\lambda})$. We choose an ω–compatible complex structure J on V and denote $g = \omega(\_, J\_)$ and by Φ^* the g–adjoint of Φ. A straightforward computation shows that $\Phi^* = J^{-1}\Phi^{-1}J$. It follows that

$$m(\lambda, \Phi) = m(\lambda, \Phi^*) = m(\lambda, \Phi^{-1}) = m(\lambda^{-1}, \Phi)$$

Hence $\sum_{\lambda \in \mathbb{C} \setminus \pm 1} m(\lambda) \in 2\mathbb{Z}$. By Corollary 2.40 the determinant of Φ is 1. It follows that $m(-1, \Phi)$ is even and therefore also $m(1, \Phi)$. $\square$

3 Symplectic Manifolds

After building some intuition about linear symplectic geometry we will now embark into the study of symplectic manifolds. An important example is the cotangent bundle equipped with the canonical symplectic form. We will encounter important classes of submanifolds, namely symplectic, (co-)isotropic and Lagrangian submanifolds. Some hightlights include Darboux' and Weinstein's theorems and the classification of closed symplectic manifolds.

The first result states that every point in a symplectic manifold has an open neighbourhood symplectomorphic to a ball in $\mathbb{R}^{2n}$. This implies that there are no local symplectic invariants in contrast to Riemannian geometry.

Weinstein's theorem says that for every closed Lagrangian submanifold L of a symplectic manifold there exist open neighbourhoods U of L and V of the zero section of T^*L and a symplectomorphism between U and V that fixes L pointwise. This means that locally all Lagrangian submanifolds look the same.

The proofs of these results are based on Moser isotopy. This technique produces a time–dependent flow that pulls back a given time dependent symplectic form ω_t to the form at time 0. This works on closed symplectic manifolds if the de Rham cohomology class of ω_t is constant in t.

Regular coisotropic submanifolds give rise to symplectic quotients. Such quotients correspond to reduced Hamiltonian systems in physics. A large class of examples arises from Hamiltonian Lie group actions. These include the complex Grassmannians, in particular complex projective space $\mathbb{C}P^n$ together with the Fubini–Studi form.

Hamilton's variational principle says that the time–evolution of a mechanical system is a path in phase space that is a critical point with fixed endpoints of the action. Via the Legendre transformation, such a critical path corresponds to a solution of Hamilton's equations on the cotangent bundle of configuration space.

3.1 Definition, Examples, Contangent bundle

Definition 3.1. Let M be a manifold. A symplectic form on M is a closed and non–degenerate 2–form on M.

Remark 3.2. If M carries a symplectic form, then it has even dimension and is orientable. The first statement follows from the classification of symplectic vector spaces, the second from Linear Liouville.

Example 3.3.

- Coordinate space $\mathbb{R}^{2n}$ together with the standard form $\omega_0 = \mathrm{d}q^i \wedge \mathrm{d}p_i$ where $q^1, p_1, \ldots, q^n, p_n$ are the canonical coordinates.
- Let $\Sigma \subset \mathbb{R}^3$ be an oriented surface. The given orientation on Σ is described by a unit normal vector field $\nu \colon \Sigma \to \mathbb{R}^3$. We define the 2–form ω on Σ by $\omega_x(v, w) = \langle \nu_x, v \times w \rangle$ for all $x \in \Sigma$, $v, w \in T_x\Sigma$.

Remark 3.4. Let (M, ω) and (M', ω') be symplectic manifolds. We denote by $\pi\colon M \times M' \to M$ and $\pi'\colon M \times M' \to M'$ the projections. Then $\widetilde{\omega} = \pi^*\omega + \pi'^*\omega'$ is a symplectic form on $M \times M'$.

The cotangent bundle of a manifold is equipped with a canonical symplectic form. It is given as follows:

Definition 3.5. Let X be a manifold. Denote by $\pi\colon T^*X \to X$ the canonical projection. The *tautological 1–form* or *canonical 1–form* or *Liouville 1–form* λ^{can} on T^*X is the smooth 1–form defined by

$$\lambda_x^{\mathrm{can}} = x \circ \mathrm{d}_x\pi \in T_x^*(T^*X) \qquad \forall x \in T_{\pi(x)}^*X,$$

where one regards x and λ_x^{can} as linear functionals on $T_{\pi(x)}X \to \mathbb{R}$ and $T_x(T^*X) \to \mathbb{R}$, respectively.

The *canonical 2–form* is defined to be

$$\omega^{\mathrm{can}} = -\,\mathrm{d}\lambda^{\mathrm{can}}.$$

The canonical 1–form is characterized by $\alpha^*\lambda^{\mathrm{can}} = \alpha$ for all $\alpha \in \Omega^1(X)$.

Example 3.6. Let W be a vector space. We canonically identify T^*W with $W \times W^*$. We denote by $\mathrm{pr}\colon W \times W^* \to W$ and $\mathrm{pr}'\colon W \times W^* \to W^*$ the projections. Let $x = (q, p) \in T^*W = W \times W^*$. The canonical projection $\pi\colon T^*W \to W$ agrees with pr. This map is linear and therefore coincides with its differential at x,

$$\mathrm{pr} = \mathrm{d}_x\pi\colon T_x(W \times W^*) = W \times W^* \to T_qW = W.$$

Hence, the canonical 1–form is given by $\lambda_x^{\mathrm{can}} = x \circ (\{q\} \times \mathrm{pr}) = p \circ \mathrm{pr} = \mathrm{pr}'(x) \circ \mathrm{pr}$.

To describe the canonical 2–form, we fix vectors $v_i = (w_i, \varphi_i) \in W \times W^*$. It follows from the definition of the exterior derivative that

$$\begin{aligned}\omega_x^{\mathrm{can}}(v_1, v_2) &= -(\,\mathrm{d}\lambda^{\mathrm{can}})_x(v_1, v_2) = \mathrm{d}(\lambda^{\mathrm{can}}v_1)(x)v_2 - \mathrm{d}(\lambda^{\mathrm{can}}v_2)(x)v_1 \\ &= (\,\mathrm{d}\,\mathrm{pr}'(x)v_2)v_1 - (\,\mathrm{d}\,\mathrm{pr}'(x)v_1)v_2 = \varphi_2(w_1) - \varphi_1(w_2) = \omega_W(v_1, v_2).\end{aligned}$$

Here ω_W denotes the canonical linear symplectic form on $W \times W^*$. Consider the case $W = \mathbb{R}^n$. We denote by $q^1, \ldots, q^n, p_1, \ldots, p_n\colon \mathbb{R}^n \times (\mathbb{R}^n)^* = \mathbb{R}^{2n} \to \mathbb{R}$ the coordinates. We have $\lambda^{\mathrm{can}} = \sum p_i\,\mathrm{d}q^i$, $\omega^{\mathrm{can}} = \sum\,\mathrm{d}q^i \wedge \mathrm{d}p_i$. The description of the canonical 1–form and 2–form in the example carries over to local formulas for these forms on a general cotangent bundle: Let X and X' be manifolds and $\varphi\colon X \to X'$ a diffeomorphism. Define the pushforward map

$$\Phi = \varphi_*\colon T^*X \to T^*X \qquad \varphi_*(q, p) = (\varphi(q), p \circ \mathrm{d}_x\varphi^{-1}).$$

Proposition 3.7. *This map satisfies* $\Phi^*\lambda_{X'}^{\mathrm{can}} = \lambda_X^{\mathrm{can}}$

Proof. Fix a point $(q, p) \in T^*X$. We write $x' = (q', p') = \Phi(x)$ and compute $(\Phi^*\lambda_{X'}^{\mathrm{can}})_x = (\lambda_{X'}^{\mathrm{can}})_{x'}\,\mathrm{d}\Phi(x) = p'\,\mathrm{d}\pi'(x')\,\mathrm{d}\Phi(x) = p\,\mathrm{d}_q\varphi^{-1}\,\mathrm{d}_q\varphi\,\mathrm{d}_q\pi = (\lambda_X^{\mathrm{can}})_x$. □

It follows from this result that $\Phi^*\omega^{\text{can}}_{X'} = -\,\mathrm{d}(\Phi^*\lambda^{\text{can}}_{X'}) = -\,\mathrm{d}\lambda^{\text{can}}_X = \omega^{\text{can}}_X$. This means that Φ is a symplectomorphism $(T^*X, \omega^{\text{can}}_X) \to (T^*X', \omega^{\text{can}}_{X'})$.

Definition 3.8. A *symplectomorphism* between two symplectic manifolds $(M, \omega) \to (M', \omega')$ is a diffeomorphism $\varphi\colon M \to M'$ such that $\varphi^*\omega' = \omega$. In physics a symplectomorphism is called a *canonical transformation*.

Remark 3.9. The map $\Phi = \varphi_*$ as above is a non–linear version of the linear push–forward introduced in the linear part.

Corollary 3.10. *The canonical 2–form ω^{can} is symplectic.*

Proof. The 2–form ω^{can} is closed since $\mathrm{d}^2 = 0$. Let $\varphi\colon U \to \mathbb{R}^n$ be a coordinate chart for X, $V = \varphi(U)$. Denote by $\Phi = \varphi_*\colon T^*U \to T^*V \subset \mathbb{R}^{2n}$ the push–forward as above. By $\Phi^*\omega^{\text{can}}_{X'} = \omega^{\text{can}}_X$ and $\omega^{\text{can}}_{\mathbb{R}^n} = \sum \mathrm{d}q^i \wedge \mathrm{d}p_i$ we have $\omega^{\text{can}}_X|_U = \Phi^*\omega_0$. Because ω_0 is non–degenerate, so is $\omega^{\text{can}}_X|_U$. □

Let $\varphi\colon U \to \mathbb{R}^n$ be a local coordinate chart for X. We denote $\Phi = \varphi_*$ and define $q^i_\varphi = \varphi^i \circ \Phi, p^\varphi_i = p_i \circ \Phi\colon T^*U \to \mathbb{R}$. These functions are called the canonical coordinates on T^*U induced by φ. They are given by

$$q^i_\varphi(q,p) = \varphi(q)^i \quad p^\varphi_i(q,p) = (p\ \mathrm{d}\varphi(q)^{-1}) = p\,\partial_i(\varphi^i)(\varphi(q)).$$

By Proposition 3.7 and the equations $\lambda^{\text{can}}_{\mathbb{R}^n} = \sum p_i\,\mathrm{d}q^i$, $\omega^{\text{can}}_{\mathbb{R}^n} = \omega_0$ we have

$$\begin{aligned}
\lambda^{\text{can}}_X|_U &= \Phi^*\left(\sum p_i\,\mathrm{d}q^i\right) = \sum (p_i \circ \Phi)\,\mathrm{d}(q^i \circ \mathrm{d}\Phi) = \sum p^\varphi_i\,\mathrm{d}q^i_\varphi \\
\omega^{\text{can}}_X|_U &= \Phi^*\left(\sum \mathrm{d}q^i \wedge \mathrm{d}p_i\right) = \sum \mathrm{d}q^i_\varphi \wedge \mathrm{d}p^\varphi_i
\end{aligned}$$

The following exercise shows that the characterisation of the canonical 2–form for the cotangent bundle of a vector space carries over to the general setting if we consider the forms along the zero section.

Exercise 3.11. Let X be a manifold with a vector bundle $E \to X$. For each point $q \in X$ find a canonical isomorphism $T_{(q,0)}E \cong T_qX \times E_q$. show that under this isomorphism the canonical 2–form on $E = T^*X$ at $(q, 0)$ agrees with the canonical linear symplectic form.

Proposition 3.12. *The form λ^{can} is uniquely determined by $\alpha^*\lambda^{\text{can}} = \alpha$ for all $\alpha \in \Omega^1(X)$.*

Proof. Let $v \in T_xX$. Then $\alpha^*\lambda^{\text{can}}(v) = \lambda^{\text{can}}_{\alpha(x)}(\,\mathrm{d}_x\alpha(v)) = \alpha_x(\,\mathrm{d}_{\alpha(x)}\pi(\,\mathrm{d}_x\alpha(v))) = \alpha_x(v)$ since $\pi \circ \alpha = \mathrm{id}$. So $\alpha^*\lambda^{\text{can}} = \alpha$. □

3.2 Classical Mechanics

In subsection 1.2 we saw that the movement of a particle in a conservative force field can be described by Hamilton's equations. The Hamiltonian function is the sum of the kinetic energy $|p|^2/2m$ and some potential energy. With ω^{can} on T^*X at hand, we can describe a general mechanical system using the same formalism. We will see that Hamilton's equations on a cotangent bundle of a manifold is equivalent to Hamilton's principle of stationary action, via the Legendre transform. This principle says that the physical path of a mechanical system is a stationary point (or critical point) of the action (with fixed end points). This principle has been generalised far beyond classical mechanics. It is fundamental for physics.

The time evolution of a mechanical system is governed by Hamilton's principle of stationary action: Such a system corresponds to a manifold X with a smooth function $L\colon \mathbb{R} \times TX \to \mathbb{R}$. X is called the *configuration space* of the system. Points in X correspond to the possible (generalised) positions of the system. L is called the *Lagrangian* of the system. Let $a \leq b$ be real numbers and $q\colon [a,b] \to X$ be a smooth path.

Definition 3.13. We define the action of q (with respect to L) as the integral

$$S(q) \coloneqq \int_a^b L(t, q(t), \dot{q}(t))\ \mathrm{d}t.$$

Hamilton's principle states that every path that occurs in nature is a 'critical point' of S among all paths with the same endpoints. This means that for every smooth family of paths $\mathbb{R} \times [a,b] \to X, (s,t) \mapsto q_s(t)$ satisfying $q_0 = q$, $q_s(a) = q(a)$, $q_s(b) = q(b)$ we have $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}\, S(q_s) = 0$.

Consider $X = \mathbb{R}^n$.

Proposition 3.14. *A path $q\colon [a,b] \to \mathbb{R}^n$ is a 'critical point' of S if and only if it satisfies the Euler–Lagrange equations:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial v^i} L(t, q(t), \dot{q}(t)) = \frac{\partial}{\partial q^i} L(t, q(t), \dot{q}(t)) \qquad \forall t \in [a,b]$$

Proof. We denote $\partial_q L = (\partial_{q^1} L, \ldots, \partial_{q^n} L)$. Assume that q is a critical point of S. Let $(s,t) \mapsto q_s(t)$ be a smooth variation of q (with fixed end points). We have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s} S(q_s) &= \int_a^b \partial_s (L(t, q_s(t), \dot{q}_s(t))\ \mathrm{d}t \\
&= \int_a^b (\partial_q L)\partial_s q_s(t) + (\partial_v L)\partial_s \partial_t q_s(t)\ \mathrm{d}t \\
&= (\partial_v L)\partial_s q_s(t)\Big|_{t=a}^{b} + \int_a^b \Big(\partial_q L - \frac{\mathrm{d}}{\mathrm{d}t}(\partial_v L)\Big)\, \partial_s q_s(t)\ \mathrm{d}t
\end{aligned}$$

Since $q_s(t) = q(t)$ for $t \in \{a,b\}$, the boundary term vanishes. Let $w\colon [a,b] \to \mathbb{R}^n$ be a smooth map satisfying $w(t) = 0$, for $t \in \{a,b\}$. We define $q_s(t) \coloneqq q(t) + sw(t)$ for all $s \in \mathbb{R}$, $t \in [a,b]$. It follows that

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=0} S(q_s) = \int_a^b (\partial_q L(t, q(t), \dot{q}(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\Big(\partial_v L(t, q(t), \dot{q}(t))\Big) w(t)\ \mathrm{d}t$$

Since this holds for every w as above it follows that

$$\left(\partial_q L - \frac{\mathrm{d}}{\mathrm{d}t}(\partial_v L)\right)(t, q(t), \dot{q}(t)) = 0.$$

Conversely, assume that the Euler–Lagrange equations are satisfied. Let $\mathbb{R} \times [a,b] \to \mathbb{R}^n, (s,t) \mapsto q_s(t)$ be a smooth variation of q. Reversing the above calculation, it follows that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} S(q_s) = 0$. Hence q is a 'critical point' of S. □

Example 3.15. Consider a particle of mass m in $\mathbb{R}^n$, subject to a conservative force $F_t := -\nabla V_t$ where $V_t \colon \mathbb{R}^n \to \mathbb{R}$ is the potential energy of the particle. Up to a constant, the Lagrangian for this system is given by

$$L(t,q,v) = \frac{m}{2}|v|^2 - V_t(q).$$

To see this, note that the Euler–Lagrange equations for this function are

$$m\ddot{q}(t)^T = \frac{\mathrm{d}}{\mathrm{d}t}\partial_v L(t,q(t),\dot{q}(t)) = \partial_q L(t,q(t),\dot{q}(t)) = -\nabla V_t(q(t)) = F_t(q(t))^T.$$

Exercise 3.16. Let $k \in \mathbb{N}$ and consider a system of k particles in $\mathbb{R}^n$ that interact through conservative central forces. Write down a Lagrangian for this system, whose Euler–Lagrange equations are Newton's second law.

The following section on holonomic constraints shows the power of Hamilton's principle. It is based on lecture notes by Kai Cieliebak.

Consider a particle in $\mathbb{R}^n$ whose motion is constrained to a submanifold $X \subset \mathbb{R}^n$. (As an example, consider the particle $S' \subset \mathbb{R}^2$.) Such constraints are called holonomic. Assume that particle is subject to an external force $F \colon [a,b] \times X \to \mathbb{R}^n$. The particle is kept inside X by a constraint force $F^{\mathrm{constr}} \colon [a,b] \to \mathbb{R}^n$. This force depends on t and on the path $q \colon [a,b] \to X$ of the particle. Then Newton's second law states

$$m\ddot{q}(t) = F(t,q(t)) + F^{\mathrm{constr}}(t) \qquad \forall t \in [a,b]$$

We assume d'Alembert's principle: The constraint force $F^{\mathrm{constr}}(t)$ is perpendicular to X. This principle means that no work is done by the constraint force.

We assume that there exists a map $f \colon \mathbb{R}^n \to \mathbb{R}^k$ such that 0 is a regular value of f and $X = f^{-1}(0)$. Then the motion of the particle can be described as follows: Assume that $q \colon [a,b] \to X$ is a smooth solution of $m\ddot{q}(t) = F(t,q(t)) + F^{\mathrm{constr}}(t)$ and that d'Alambert's principle holds. We denote $k := \operatorname{codim} X = n - \dim X$, and define $\lambda \colon [a,b] \to (\mathbb{R}^k)^*$ to be the unique map satisfying $F^T_{\mathrm{constr}} = \lambda Df$. The components $\lambda_i(t)$ can be interpreted as Lagrange multipliers. We denote by $D^2 f(q) \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ the Hessian defined by $D^2 f(q)(w,w') = \sum_{i,j=1}^n \partial_i \partial_j f(q_0) w^i w^j$. Taking two time derivatives of the equations $f \circ q = 0$, we obtain $Df(q)\dot{q} = 0$ and

$$0 = (D^2 f)_q(\dot{q},\dot{q}) + Df(q)\ddot{q} = (D^2 f)_q(\dot{q},\dot{q}) + \frac{1}{m}\Big(Df(q)F(t,q) + Df(q)Df(q)^T \lambda(t)^T\Big). \quad (*)$$

Since by assumption 0 is a regular value of f, $Df(q(t)) \colon \mathbb{R}^n \to \mathbb{R}^k$ is surjective and hence the $k \times k$ matrix $Df(q(t))Df(q(t))^T$ is invertible for every t. Hence $\lambda(t)$ is determined by $(*)$.

Furthermore, $\lambda(t)$ depends only on the point $(t,q(t),\dot{q}(t)) \in [a,b] \times TX$, rather than on the whole path $q \colon [a,b] \to X$. It follows that the constraint force is given by a map

$F_{\text{constr}}\colon [a,b]\times TX \to \mathbb{R}^n$. We may compute the constraint force by solving $(*)$ for λ and plugging the solution into $F_{\text{constr}}^T = \lambda Df$. The motion of the system is then determined by Newton's equation with $F_{\text{constr}}(t) = F^{\text{constr}}(t, q(t), \dot{q}(t))$.

Example 3.17 (Centripetal force)**.** Suppose $m = 1$, $X := S^{n-1} \subset \mathbb{R}^n$, $F = 0$. To determine the motion of the particle, we may choose $f\colon \mathbb{R}^n \to \mathbb{R}, f(q) = |q|^2/2$. We have $Df(q) = q^T$ and $(D^2 f)_q(w, w') = w \cdot w'$. Therefore, it follows fromm $(*)$ that $F_{\text{constr}}(t, q, v) = -|v|^2 q$. Therefore, Newton's equation $m\ddot{q} = F + F^{\text{constr}}$ reads $\ddot{q} = F_{\text{constr}} = -|\dot{q}|^2 q$. The unique solution of this ODE with initial condition $(q(0), \dot{q}(0)) = (q_0, v_0) \in TS^{n-1}$ is given by

$$q(t) = q_0 \cos(|v_0|t) + \frac{v_0}{|v_0|}\sin(|v_0|t).$$

Exercise 3.18. Show that the constraint force is given by $F_{\text{constr}}(t, q, v) = m\,\mathrm{II}_q(v, v) - F^{\perp}(t, q)$ where II_q is the second fundamental form at q and $F^{\perp}$ is the component of the external force F perpendicular to X.

Now we reformulate the restrained system in terms of the Lagrangian formalism. Consider the case of a conservative external force $F_t = -\nabla V_t$. Then a simpler approach to finding the motion of the restrained particle is the following: We define the Lagrangian

$$L\colon [a,b]\times TX \to \mathbb{R}, L(t, q, v) := \frac{m}{2}|v|^2 - V_t(q)$$

Proposition 3.19. *Let $q\colon [a,b] \to X$ be a smooth path in X. The following are equivalent:*

1. *The path q satisfies Newton's equation $m\ddot{q}(t) = F_t + F^{\text{constr}}(t)$ for some constraint force $F^{\text{constr}}(t)$ perpendicular to $T_q X$.*
2. *q is a critical point of the action $S(q) = \int_a^b L(t, q(t), \dot{q}(t))\ \mathrm{d}t$ among variations in X with fixed and points. This means that $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} S(q_s) = 0$ for every smooth family of paths $\mathbb{R}\times[a,b] \to X, (s,t) \mapsto q_s(t)$ satisfying $q_0 = q$, $q_s(a) = q(a)$, $q_s(b) = q(b)$.*

Proof. Assume that the second property holds. As in the proof of Proposition 3.14 differentiation under the integral sign and integration by parts yields

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\bigg|_{s=0} S(q_s) = \int_a^b (\partial_q L - \frac{\mathrm{d}}{\mathrm{d}t}\partial_v L)\, w(t)\ \mathrm{d}t$$

for every smooth map $w\colon [a,b] \to \mathbb{R}^n$ satisfying $w(t) \in T_{q(t)}X$ for all $t \in [a,b]$ and $w(a) = w(b) = 0$. It follows that $-\nabla V_t - m\ddot{q} = \partial_q L - \frac{\mathrm{d}}{\mathrm{d}t}(\partial_v L) \in T_{q(t)}X^{\perp}$. The statement follows. The other implication follows from a similar argument. □

Corollary 3.20. *Let $q\colon [a,b] \to X$ be smooth. q satisfies Newton's equations with $F_{\text{constr}}(t) \perp T_{q(t)}X$ if and only if q satisfies the Euler Lagrange equation in local coordinates.*

Proof. Assignment 7, based on Proposition 3.19.

Let (X, g) be a Riemannian manifold, $a < b$. Recall that a geodesic on the interval $I = [a, b]$ is a smooth path $q\colon I \to X$ which is locally length minimizing and for which $\|\dot{q}\|$ (induced by g) is constant. This means, for all $t_0 \in I$ there exists a neighbourhood $t_0 \in U \subset I$ such that

$$\ell(q) := \int_t^{t'} \|\dot{q}\| \, \mathrm{d}t = d(q(t), q(t'))$$

for all $t < t' \in U$. Here d denotes the distance function induced by g. We define the Lagrangian $L\colon TX \to \mathbb{R}$ by $L(q, v) = \frac{1}{2}\|v\|^2 = \frac{1}{2}g(v, v)$ with the corresponding action $S(q) = \frac{1}{2}\int_a^b \|\dot{q}\| \, \mathrm{d}t$.

Exercise 3.21 (Assignment 7)**.** Show that every non–constant geodesic is a critical point of S. Find the Euler–Lagrange equations. (The converse is also true.)

The Legendre transform allows us to reformulate Lagrangian mechanics as Hamiltonian mechanics and vice–versa. The idea is that the Legendre transform replaces the velocity variable $v = \dot{q}$ by the momentum p and maps the Lagrangian L to the Hamiltonian H. The Euler–Lagrange equation is equivalent to Hamiltons equation for H. Heuristically, we define the Legendre transform of L by setting $p := \partial_v L(q, v) = (\partial_{v^1} L, \ldots, \partial_{v^n} L)(q, v)$ and defining $H = L^*$ by $H(q, p) = pv(q, p) - L(q, v(q, p))$. Here we view v as a function of q and p.

Example 3.22. Consider a non–relativistic particle in $\mathbb{R}^n$ of mass m in a conservative potential U. As we saw its Lagrangian is $L = T - U\colon \mathbb{R}^{2n} \to \mathbb{R}$, where $T := \frac{1}{2}m|v|^2$. We saw that the Hamiltonian is $H = T + U\colon \mathbb{R}^{2n} \to \mathbb{R}$. This corresponds to $\partial_v L(q, v) = mv^T$. Assume that L is smooth. The *Legendre condition* at a point $(q_0, v_0) \in \mathbb{R}^{2n}$ states that the Hessian of the map $L(q_0, \_)\colon \mathbb{R}^n \to \mathbb{R}$ at v_0 is non–degenerate. This means that $\det((\partial_{v^i}\partial_{v^j} L(q_0, v_0))_{i,j}) \neq 0$. If this is true then v can locally be expressed as a smooth function of q and p.

Proposition 3.23. *If L satisfies the Legendre condition at (q_0, v_0) then there exist open neighbourhoods $U, V, W \subset \mathbb{R}^n$ of $q_0, v_0, p_0 := \partial_v L(q_0, v_0)$ and a smooth fuction $\mathbf{v}\colon U \times W \to V$ such that $(\partial_v L)^{-1}(p) \cap (U \times V) = \mathrm{gr}(\mathbf{v}(\_, p)) = \{(q, \mathbf{v}(q, p))\colon q \in U\}$ for all $p \in W$.*

Proof. Assignment 8.

Remark 3.24. This means that for all $p \in W$ we have $\partial_v L(q, \mathbf{v}(q, p)) = p$ and for all $(q, p) \in U \times W$ there is at most one $v \in V$ such that $\partial_v L(q, v) = p$.

Remark 3.25. The Legendre condition is satisfied for $L = T - U$ with $T(q, v) = \frac{1}{2}m|v|^2$.

The derivative $p := \partial_v L(q, v)$ is called the *canonical momentum* associated to L. We now define the Legendre transform in a more general setting: For a Lagrangian L on the tangent bundle of some manifold, which is fibrewise super–linear and locally stongly convex.

Let V be a vector space and $f\colon V \to \mathbb{R}$ a function.

Definition 3.26. f is *super–linear* (at ∞) if for all $p \in V^*$ we have

$$\inf_{v \in V} \big(f(v) - pv \big) > -\infty.$$

Definition 3.27. We define the *Legendre transform* of f to be

$$f^* \colon V^* \to \mathbb{R}, \qquad p \mapsto \sup_{v \in V} \big(pv - f(v) \big)$$

Example 3.28 (Assignment 8)**.** Let $a \in (1, \infty)$ and $f \colon \mathbb{R} \to \mathbb{R}$ defined by $f(v) \coloneqq |v|^a/a$. Then f is super–linear and $f^*(p) = |p|^{a'}/a'$ where $1/a' + 1/a = 1$.

Definition 3.29. We call a smooth function $f \colon V \to \mathbb{R}$ *locally strongly convex* if its Hessian at any point in V is positive definite.

Example 3.30. Let $\langle \_, \_ \rangle$ be an inner product on V. The function $f \colon V \to \mathbb{R}$ defined by $f(v) \coloneqq \|v\|^2 = \langle v, v \rangle$ is locally strongly convex and super–linear. For $k = 2, 3, \ldots$ the function $v \mapsto \|v\|^{2k}$ is super–linear, but not locally strongly convex. The function $f \colon \mathbb{R} \to \mathbb{R}$ given by $f(v) \coloneqq e^v$ is locally strongly convex, but not super–linear.

Let now X be a manifold and $L \colon TX \to \mathbb{R}$ be smooth. Assume that L is fibrewise super–linear. This means that for every $q \in X$, the function $L(q, \_) \colon T_qX \to \mathbb{R}$ is super–linear.

Definition 3.31. We define the *Legendre transform* of L to be $H \coloneqq L^* \colon T^*X \to \mathbb{R}$, which is defined by $H(q,p) = L(q, \_)^*(p)$.

Definition 3.32. We define the *fibre derivative* of L at $(q, v) \in TX$ to be the covector $p \coloneqq \partial_v L(q, v) \in T_q^*X$ given by

$$\partial_v L(q,v) w \coloneqq D(L(q, \_))(v) w = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} L(q, v + tw)$$

for all $w \in T_qX = T_v(T_qX)$.

Definition 3.33. For a symplectic manifold (M, ω) and a smooth function $H \colon M \to \mathbb{R}$ we define the *Hamiltonian vector field* generated by H to be the unique vector field $X_H = X_H^{\omega}$ such that $\mathrm{d}H = \omega(X_H, \_)$.

Theorem 3.34 (Legendre transform)**.** *Let X be a manifold and $L \colon TX \to \mathbb{R}$ a smooth function that is fibrewise locally strongly convex and superlinear. Then*

1. *The map*
$$TX \mapsto T^*X, \qquad (q, v) \mapsto (q, \partial_v L(q, v))$$
is a diffeomorphism.

2. *We define*
$$\mathbf{v} \colon T^*X \to TX, \qquad (q,p) \mapsto (\partial_v L(q, \_))^{-1}(p).$$
Then $L^*(q,p) = p\mathbf{v}(q,p) - L(q, \mathbf{v}(q,p))$ for all $(q,p) \in T^*X$.

3. *The Legendre transform is involutive: The function L^* is smooth and fibrewise locally strongly convex and superlinear and satisfies $(L^*)^* = L$.*

4. *A smooth path $q\colon [a,b] \to X$ is a critical point of the action*

$$S(q) = \int_a^b L(q, \dot{q})\ \mathrm{d}t$$

if and only if the path

$$x\colon [a,b] \to T^*X, \qquad x = (q, \partial_v L(q,\dot{q}) \circ \dot{q})$$

solves Hamilton's equations $\dot{x} = X_{L^*}^{\omega^{\text{can}}} \circ x$.

The proof of Theorem 3.34 is based on

Proposition 3.35. *Let V be a vector space, $f\colon V \to \mathbb{R}$ a smooth, locally strongly convex and superlinear function and $p_0 \in V^*$. Then*

1. *The function*
$$g = f_{p_0}\colon V \to \mathbb{R}, \qquad f_{p_0}(v) = p_0 v - f(v)$$
has a unique critical point.

2. *The function attains its maximum at this critical point.*

3. *We define $\mathbf{v}\colon V^* \to V$ such that $\mathbf{v}(p)$ is the unique critical point of f_p. This map is an inverse for $\mathrm{d}f\colon V \to V^*$.*

4. *The Legendre transform $f^*\colon V^* \to \mathbb{R}$ is superlinear, smooth and locally strongly convex and $\mathrm{d}(f^*) = \iota \circ (df)^{-1}$ and $f^{**} \circ \iota = f$ hold, where $\iota\colon V \to V^{**}$ is the canonical isomorphism.*

Proof of Theorem 3.34. Applying the third part of Proposition 3.35 to the functions $L(q, \_)$ for $q \in X$, the map $(q,v) \to (q, \partial_v L(q,v))$ is bijective with inverse $(q,p) \mapsto (q, \mathbf{v}(q,p))$. By our hypothesis, the Hessian of $L(q,\_)\colon T_qX \to \mathbb{R}$ at v is positive definite and thus non–degenerate for every $(q,v) \in TX$. Then Proposition 3.23 implies that $\mathbf{v}$ is smooth. Hence the same holds for the map $(q,p) \mapsto (q, \mathbf{v}(q,p))$. This proves the first part of Theorem 3.34.

The second part follows from the first three parts of Proposition 3.35 and the third follows from the last part of Proposition 3.35. We now prove the fourth part.

Consider the case $X = \mathbb{R}^n$ and let $H = L^*\colon \mathbb{R}^n \times (\mathbb{R}^n)^* \to \mathbb{R}$. By the definition of $\mathbf{v}$ we have $\partial_v L(q, \mathbf{v}(q,p)) = p$ and therefore

$$\partial_q H(q,p) = \partial_q(p\mathbf{v} - L(q,\mathbf{v})) = p\partial_q \mathbf{v} - \partial_v L(q,\mathbf{v})\partial_q \mathbf{v} - \partial_q L(q,\mathbf{v}) = -\partial_q L(q,\mathbf{v}).$$

Assume that $q\colon [a,b] \to \mathbb{R}^n$ is a critical point of S. We define $p \coloneqq \partial_v L(q,\dot{q})$ and $x = (q,p)\colon [a,b] \to T^*\mathbb{R}^n$. Using the fourth part of Proposition 3.35 it follows that

$\dot{q} = (\partial_v L(q, \_)^{-1})(p) = \partial_p H(q,p)$. This is the first of Hamilton's equations. Since q is a critical point of S it satisfies the Euler–Lagrange equation

$$\dot{p} = \frac{\mathrm{d}}{\mathrm{d}t}\partial_v L(q,\dot{q}) = \partial_q L(q,\dot{q}) = -\partial_q H(q,p).$$

We have $\omega^{\mathrm{can}}_{\mathbb{R}^n} = \omega_0$. Threfore this is equivalent to $\dot{x} = X_H^{\omega^{\mathrm{can}}} \circ x$. The proof of the reverse implication and in the general setting is similar. □

Proof of Proposition 3.35. Since f is superlinear, we have

$$\sup_{v\in V}(g(v) + pv) = \sup_{v\in V}\big((p_0+p)v - f(v)\big) < \infty$$

for all $p \in V^*$. It follows that for every $C \in \mathbb{R}$ there exists a compact subset $K \subset V$ such that $g(v) \leq C$ for all $v \in V \setminus K$. We choose K as above for $C = \sup_{v\in V} g(v) - 1$. It follows that g attains its maximum at some point $v_0 \in K$. By Assignment 8 the function has no other critical point. This proves the first part of Proposition 3.35.

To prove the third part let $v \in V$ and set $p = \mathrm{d}f(v)$. We have $\mathrm{d}f_p(v) = p - \mathrm{d}f(v) = 0$ and therefore $\mathbf{v}(p) = v$. This means that $\mathbf{v} \circ (\mathrm{d}f) = \mathrm{id}_V$. On the other hand, for every $p \in V^*$ we have

$$0 = (\mathrm{d}f_p) \circ \mathbf{v}(p) = p - (\mathrm{d}f) \circ \mathbf{v}(p)$$

and therefore $(\mathrm{d}f) \circ \mathbf{v} = \mathrm{id}$.

For the fourth part, superlinearity follows from Assignment 8 and smoothness from the inverse function theorem. Let $p \in V^*$. The equality $f^*(p) = p\mathbf{v}(p) - f(\mathbf{v}(p))$ implies that

$$\mathrm{d}(f^*)(p) = \mathbf{v}(p) + (p - \mathrm{d}f(\mathbf{v}(p)))\,\mathrm{d}\mathbf{v}(p).$$

By the third part we have $p - \mathrm{d}f(\mathbf{v}(p)) = 0$, so $\iota \circ \mathrm{d}(f^*) = (\mathrm{d}f)^{-1}$. Let $v \in V$. We write $p = (\mathrm{d}(f^*)^{-1} \circ \iota(v)) \in V^*$. Since f^* is smooth, superlinear and locally strongly convex, it follows that

$$f^{**}(\iota(v)) = \sup_{p'\in V^*}\big(\iota(v)p' - f^*(p')\big) = \iota(v)p - f^*(p) = pv - (p(\mathrm{d}f)^{-1}(p) - f \circ (\mathrm{d}f)^{-1}(p)),$$

which is just $f(v)$ since $(\mathrm{d}f)^{-1}(p) = ((\mathrm{d}f)^{-1} \circ (\mathrm{d}f^*)^{-1} \circ \iota)(v) = v$. □

3.3 Symplectomorphisms, Hamiltonian diffeomorphisms and Poisson brackets

Let (M,ω) and (M',ω') be symplectic manifolds. (In this section we only consider manifolds without boundary.) Recall that a symplectomorphism between M and M' is a diffeomorphism $\varphi\colon M \to M'$ such that $\varphi^*\omega' = \omega$. We denote by $\mathrm{Symp}(M,\omega)$ the set of all symplectomorphisms $(M,\omega) \to (M,\omega')$. This is a subgroup of the group of diffeomorphisms of M, denoted $\mathrm{Diff}(M)$.

Definition 3.36. We call a vector field X on M *symplectic* if the one–form $\iota_X\omega = \omega(X, \_)$ is closed. We denote by $\mathfrak{X}(M,\omega)$ the vector space of symplectic vector fields on M.

Formally, $\mathrm{Symp}(M,\omega)$ is an infinite dimensional subgroup of $\mathrm{Diff}(M)$ with Lie algebra $\mathfrak{X}(M,\omega)$. The following result motivates this interpretation.

Proposition 3.37 (Characterisation of symplectic isotopies)**.**

1. *Let I be an interval, $t_0 \in I$, (M,ω) a symplectic manifold and $\varphi\colon I \times M \to M$ a smooth map such that $\varphi^t = \varphi(t, \_)$ is a diffeomorphism $M \to M$ for all $t \in I$ and such that $\varphi^{t_0} \in \mathrm{Symp}(M)$. Then φ^t is a symplectomorphism for all $t \in I$ if and only if the vector field $X_t = (\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t) \circ \varphi_t^{-1}$ lies in $\mathfrak{X}(M,\omega)$ for all t.*

2. *If $X, Y \in \mathfrak{X}(M,\omega)$ then $\iota_{[X,Y]}\omega = -\,\mathrm{d}H$ for $H = \omega(X,Y)$.*

Remark 3.38. A smooth map $\varphi\colon I \times M \to M$ such that $\varphi^t \in \mathrm{Diff}(M)$ is called a *smooth isotopy.* If $\varphi^t \in \mathrm{Symp}(M,\omega)$ for all t then it is called a symplectic isotopy. If $t_0 = 0$ and $\varphi^{t_0} = \mathrm{id}$ then $(\varphi^t)_{t\in I}$ is the flow of $(X_t)_{t\in I}$. In this case X_0 is the derivative of the path $t \mapsto \varphi_t$ in the 'Lie group' $\mathrm{Diff}(M)$ at time $t = 0$.

The second part of Proposition 3.37 implies that the Lie bracket preserves $\mathfrak{X}(M,\omega)$.

Proof of Proposition 3.37. By Cartan's formula the Lie derivative of ω with respect to a vector field X is

$$\mathcal{L}_X\omega = (\,\mathrm{d}\iota_X + \iota_X\,\mathrm{d})\omega = \,\mathrm{d}\iota_X\omega.$$

Assume that $\varphi_t \in \mathrm{Symp}(M,\omega)$ for every t. Then

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^*\omega) = \frac{\mathrm{d}}{\mathrm{d}t}(\omega(\,\mathrm{d}\varphi_t\_,\,\mathrm{d}\varphi_t\_)) = \varphi_t^*\mathcal{L}_{X_t}\omega,$$

and therefore $\mathcal{L}_{X_t}\omega = 0$ for all t. It follows that $\iota_{X_t}\omega$ is closed for all t. Conversely, assume that $\iota_{X_t}\omega$ is closed for all t. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\omega = \varphi_t^*\mathcal{L}_{X_t}\omega = 0$$

for all t. Interpreting this equality at any fixed point in M with respect to time t and using the hypothesis $\varphi_{t_0} = \mathrm{id}$, we obtain $\varphi_t^*\omega = \varphi_{t_0}^*\omega = 0$ for all t.

To prove the second part, consider the Leibniz rule for Lie derivatives:

$$\begin{aligned}
\mathcal{L}_X(\iota_Y\omega(Z)) &= (\mathcal{L}_X\iota_Y\omega)(Z) + \iota_Y\omega(\mathcal{L}_X Z)\\
\mathcal{L}_X(\omega(Y,Z)) &= (\mathcal{L}_X\omega)(Y,Z) + \omega(\mathcal{L}_X Y, Z) + \omega(Y, \mathcal{L}_X Z)
\end{aligned}$$

Subtracting these, one obtains using Cartan's formula

$$0 = \mathcal{L}_X\iota_Y\omega - \iota_Y\mathcal{L}_X\omega - \iota_{\mathcal{L}_X Y}\omega = \iota_X\,\mathrm{d}\iota_Y\omega + \,\mathrm{d}\iota_X\iota_Y\omega - \iota_Y\iota_X\,\mathrm{d}\omega - \iota_Y\,\mathrm{d}\iota_X\omega - \iota_{[X,Y]}\omega.$$

Now if $X, Y \in \mathfrak{X}(M,\omega)$, the first, third and fourth terms vanish, so we get

$$\iota_{[X,Y]}\omega = \,\mathrm{d}\iota_X\iota_Y\omega = \,\mathrm{d}(\omega(Y,X)) = -\,\mathrm{d}H. \qquad \square$$

Recall that the Hamiltonian vector field generated by a smooth function $H\colon M \to \mathbb{R}$ is the unique vector field X_H satisfying $\mathrm{d}H = \omega(X_H, \_)$. Every such vector field is symplectic.

Definition 3.39. Let I be an interval containing 0 and $H\colon I \times M \to \mathbb{R}$ a smooth function. We define the *Hamiltonian flow* of H to be the flow φ_H of the time–dependent Hamiltonian vector field $(X_{H_t})_{t\in I}$ where $H_t = H(t, \_)$.

Remark 3.40. φ_H is a map from an open subset $\mathcal{D}_H \subset I \times M$ to M. It is smooth and for every $t \in I$ the map $\varphi_H^t = \varphi_H(t, \_)\colon \mathcal{D}_H^t = \{x \in M\colon (t,x) \in \mathcal{D}_H\} \to M$ is injective and a smooth immersion.

Example 3.41. Consider the Hamiltonian $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ given by $H(x) = \frac{1}{2}|x|^2 = \frac{1}{2}(|q|^2 + |p|^2)$ which describes a particle in $\mathbb{R}^n$ of mass $m = 1$ subject to the force exerted by and ideal spring. We identify $\mathbb{R}^{2n} = \mathbb{C}^n$. By Assignment 1, Exercise 5, we have $X_H(x) = -ix$ with the flow $\varphi_H\colon \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$ given by $\varphi_H^t(x) = e^{-it}x$.

Example 3.42. Consider $M = S^2 \subset \mathbb{R}^3$ with the symplectic form ω given by Example 3.3: $\omega_x(v,w) = x \cdot (v \times w)$. Look at the height function $H\colon S^2 \to \mathbb{R}$, $H(x) = x_3$. We have $X_H(x) = (-x_2, x_1, 0)$ and $\varphi_H^t(x) = (R^t(x_1,x_2), x_3)$ where $R^t\colon \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counterclockwise rotation by t.

Definition 3.43. A *Hamiltonian diffeomorphism* of M is a map $\varphi\colon M \to M$ for which there exists a smooth function $H\colon [0,1] \times M \to \mathbb{R}$ such that $\mathcal{D}_H = [0,1] \times M$ and $\varphi_H^t\colon M \to M$ is surjective for all t and $\varphi = \varphi_H^1$. We denote by $\mathrm{Ham}(M,\omega)$ the set of Hamiltonian diffeomorphisms.

Remark 3.44. It follows from Remark 3.40 that every Hamiltonian diffeomorphism is in fact a diffeomorphism of M. If H has compact support then the global definition is automoatic. This is the case if M is closed. It follows from Proposition 3.37 that every Hamiltonian diffeomorphism is a symplectomorphism.

Exercise 3.45. Find and example of a triple (M,ω,H) such that $\mathcal{D}_H \neq I \times M$. Find an example in which φ_H^t is not surjective for some t.

Definition 3.46. Let I be an interval. A *Hamiltonian isotopy* is a smooth map $\varphi\colon I \times M \to M$ such that $\varphi^{t_0} \in \mathrm{Ham}(M,\omega)$ for some t_0, φ^t is surjective for all t and there exists a function $H\colon I \times M \to \mathbb{R}$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi = X_{H_t} \circ \varphi_t$$

Remark 3.47. It follows from Remark 3.40 and Proposition 3.48 that for every Hamiltonian isotopy the map φ^t is a Hamiltonian diffeomorphism for all t.

If M is closed then conversely, if $\varphi\colon I \times M \to M$ is such that $\varphi^t \in \mathrm{Ham}(M,\omega)$ for all t, there exists a function $H\colon I \times M \to \mathbb{R}$ such that $\frac{\mathrm{d}}{\mathrm{d}t}\varphi = X_{H_t} \circ \varphi_t$. The proof of this statement uses the flux homomorphism.

Proposition 3.48.

1. *For every smooth function* $H\colon M \to \mathbb{R}$ *and* $\varphi \in \mathrm{Symp}(M,\omega)$ *we have*

$$X_{H\circ\varphi} = \varphi^* X_H = (\,\mathrm{d}\varphi)^{-1} X_H \circ \varphi.$$

2. $\mathrm{Ham}(M,\omega)$ *is a normal subgroup of* $\mathrm{Symp}(M,\omega)$*.*

Proof. The first part follows from Assignment 9. For the second part, let $\varphi, \psi \in \mathrm{Ham}(M,\omega)$. Choose smooth functions $F, G\colon [0,1] \times M \to \mathbb{R}$ such that $\varphi = \varphi_F^1$ and $\psi = \varphi_G^1$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_F^t \circ \varphi_G^t) &= X_{F^t} \circ \varphi_F^t \circ \varphi_G^t + (\,\mathrm{d}\varphi_F^t) X_{G^t} \circ \varphi_G^t \\ &= (X_{F^t} + \mathrm{d}\varphi_F^t X_{G^t} \circ (\varphi_F^t)^{-1}) \circ \varphi_F^t \circ \varphi_G^t \\ &= (X_{F^t + G^t \circ (\varphi_F^t)^{-1}}) \circ \varphi_F^t \circ \varphi_G^t\end{aligned}$$

It follows that $(\varphi_F^t \circ \varphi_G^t)$ is the Hamiltonian flow of $F^t + G^t \circ (\varphi_F^t)^{-1}$, so $\varphi \circ \psi \in \mathrm{Ham}(M,\omega)$.

The Hamiltonian flow of $-F^t \circ \varphi_F^t$ is $(\varphi_F^t)^{-1}$, so $\varphi^{-1} \in \mathrm{Ham}(M,\omega)$. Let $\varphi \in \mathrm{Ham}(M,\omega)$ and $\psi \in \mathrm{Symp}(M,\omega)$. We show $\psi^*\varphi = \psi^{-1} \circ \varphi \circ \psi \in \mathrm{Ham}(M,\omega)$. Choose a smooth function $H\colon [0,1] \times M \to \mathbb{R}$ such that $\varphi_H^1 = \varphi$ and write $H^t = H(t, \_)$. Consider the map $\widetilde{\varphi}\colon [0,1] \times M \to M$ with $\widetilde{\varphi}^t = \psi^* \varphi_H^t$. This solves the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\varphi}^t = \mathrm{d}(\psi^{-1})\frac{\mathrm{d}}{\mathrm{d}t}\varphi_H^t \circ \psi = \mathrm{d}(\psi^{-1}) X_{H^t} \circ \psi \circ \widetilde{\varphi}^t = X_{H^t \circ \psi} \circ \widetilde{\varphi}^t.$$

Hence, $\widetilde{\varphi}$ is the Hamiltonian flow of $H^t \circ \psi$. Since $\varphi_H^1 = \varphi$ it follows that $\psi^*\varphi \in \mathrm{Ham}(M,\omega)$. □

Remark 3.49. For an alternative proof that $\mathrm{Ham}(M,\omega)$ is closed under composition, given Hamiltonian isotopies φ^t and ψ^t, consider

$$\chi^t = \begin{cases} \varphi^{2t} & 0 \le t \le 1/2 \\ \psi^{2t-1} \circ \varphi^1 & 1/2 < t \le 1. \end{cases}$$

By reparametrising we may achieve that χ is smooth and the Hamiltonian flow of a time–dependent function. Since $\chi^1 = \psi^1 \circ \varphi^1$ this shows that $\mathrm{Ham}(M,\omega)$ is closed under composition.

Formally, the group $\mathrm{Ham}(M,\omega)$ is a Lie subgroup of $\mathrm{Symp}(M,\omega)$ with Lie algebra isomorphic to $C^\infty(M)/\{\text{locally constant functions}\}$.

Remark 3.50. In the 1970s Banyaga proved that $\mathrm{Ham}(M,\omega)$ is simple if M is closed and connected. The proof can be found in McDuff–Salomon, Theorem 10.25. The Lie bracket on '$\mathrm{Lie}\,\mathrm{Ham}(M,\omega)$' is formally given by the Poisson bracket: Let M be a manifold and ω a non–degenerate 2–form.

Definition 3.51. The *Poisson bracket* wrt. ω is the map

$$\{\_,\_\}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M), \qquad \{F,G\} = \omega(X_F, X_G) = \mathrm{d}F(X_G)$$

Proposition 3.52.

1. $\{\{F,G\},H\} + \{\{G,H\},F\} + \{\{H,F\},G\} = \frac{1}{2}\,\mathrm{d}\omega(X_F, X_G, X_H)$.

2. *For every pair $F, G \in C^\infty(M)$ we have $[X_F, X_G] = -X_{\{F,G\}}$.*

3. *Let $F, H \in C^\infty(M)$ such that the flow of H exists globally for all time. Then the condition $F \circ \varphi_H^t = F$ for all t is equivalent to $\{F, H\} = 0$.*

4. *Let $\varphi\colon M \to M$ be a symplectomorphism. Then φ preserves the Poisson bracket, i.e.*
$$\{F \circ \varphi, G \circ \varphi\} = \{F, G\} \circ \varphi$$

Proof. We prove the second part. Let $F, G \in C^\infty(M)$, then

$$[X_F, X_G] = \mathcal{L}_{X_F} X_G = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} D(\varphi_{X_F}^t)^{-1}(X_G) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} X_{G\circ\varphi_{X_F}^t}.$$

So for any $Y \in TM$ we have

$$\omega([X_F, X_G], Y) = \omega\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} X_{G\circ\varphi_{X_F}^t}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \omega(X_{G\circ\varphi_{X_F}^t}, Y) = Y\left(\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} G \circ \varphi_{X_F}^t\right).$$

But the inner function is at a point $p \in M$ given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} G \circ \varphi_{X_F}^t(p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} G \circ \varphi_{X_F}^p(t) = \mathrm{d}G(\dot{\varphi}_{X_F}^p(0)) = \mathrm{d}G(X_F) = \{G, F\} = -\{F, G\},$$

so $\omega([X_F, X_G], Y) = -Y\{F, G\} = -\omega(X_{\{F,G\}}, Y)$, which by non–degeneracy proves the desired statement. The other parts are proven in Assignment 9. □

Remark 3.53. Consider the case in which ω is non–degenerate, i.e. symplectic. By Proposition 3.37 the vector space $\mathfrak{X}(M, \omega)$ of symplectic vector fields is closed under the Lie bracket. Furthermore by Proposition 3.52 the Poisson bracket defines the structure of a Lie algebra und the vector space $C^\infty(M)$. By the second part of Proposition 3.52 the space $\mathrm{Ham\,Vect}(M, \omega)$ of Hamiltonian vector fields is a Lie subalgebra of $\mathfrak{X}(M, \omega)$ and the map

$$C^\infty(M) \to \mathrm{Ham\,Vect}(M, \omega), \qquad H \mapsto X_H$$

is a surjective Lie algebra homomorphism with respect to $\{\_, \_\}$ and $[\_, \_]$. The kernel of this homomorphism consists of the locally constant functions, so it induces in isomorphism

$$\mathrm{Ham\,Vect}(M, \omega) \cong C^\infty(M)/\{\text{locally constant functions}\}.$$

The third part means that the following conditions are equivalent:

1. For every solution $x\colon \mathbb{R} \to M$ of Hamilton's equation $\dot{x} = X_H \circ x$ the function $F \circ x\colon \mathbb{R} \to \mathbb{R}$ is constant.

2. F and H Poisson commute, i.e. $\{F, H\} = 0$.

A function F as in the first condition is called an *integral* or *constant of motion* of H. In particular, the fact $\{H, H\} = 0$ implies conservation of energy, i.e. $H \circ x$ is constant if x solves Hamilton's equation.

3.4 Darboux' Theorem and Moser isotopy

Theorem 3.54 (Darboux)**.** *Let (M,ω) be a $2n$–dimensional symplectic manifold and $x \in M$. There exists an open neighbourhood $U \subset M$ of x and a chart $\varphi\colon U \to \mathbb{R}^{2n}$ satisfying $\varphi^*\omega_0 = \omega$.*

Remark 3.55. This means that locally all symplectic manifolds look the same, i.e. are symplectomorphic. In contrast with this Riemannian manifolds of the same dimension need not be locally isometric.

Theorem 3.54 is a manifold version of the second part of Theorem 2.13, which implies that every symplectic vector space of dimension $2n$ is isomorphic to $(\mathbb{R}^{2n}, \omega_0)$. In contrast to Theorem 3.54 globally two symplectic forms on a manifold M may not be isomorphic. This may happen for example if $\mathrm{Vol}(M,\omega) \neq \mathrm{Vol}(M,\omega')$.

Definition 3.56. A chart $\varphi\colon U \to \mathbb{R}^{2n}$ is called a *Darboux chart* if $\varphi^*\omega_0 = \omega$.

Proposition 3.57. *Let M be a $2n$–dimensional manifold, $N \subset M$ a closed submanifold and ω^0 and ω^1 symplectic forms that agree on N. Then there exist open neigbourhoods U_0 and U_1 of N and a diffeomorphism $\psi\colon U_0 \to U_1$ such that $\psi^*\omega^1 = \omega^0$ and $\psi|_N = \mathrm{id}$.*

Proof of Theorem 3.54. We choose an open neighbourhood $\widetilde{U}$ of $x \in M$ and a chart $\widetilde{\varphi}\colon \widetilde{U} \to \mathbb{R}^{2n}$ such that $\widetilde{\varphi}(x) = 0$. By Theorem 2.13 there is an automorphism Φ of the vector space $\mathbb{R}^{2n}$ such that $\Phi^*\omega_0 = ((\widetilde{\varphi}^{-1})^*\omega)_{\varphi(x)} = \omega_x((\mathrm{d}\widetilde{\varphi})_x^{-1}\_\_, (\mathrm{d}\widetilde{\varphi})_x^{-1}\_\_)$. We define $\omega^0 = \omega|_{\widetilde{U}}$, $\omega^1 = \widetilde{\varphi}^*(\Phi^*\omega_0)$. The hypotheses of Proposition 3.57 are satisfied with $M = \widetilde{U}$ and $N = \{x\}$. Hence, there exist U and ψ as in its conclusion. The chart $\varphi = \Phi \circ \widetilde{\varphi} \circ \psi$ satisfies $\varphi^*\omega_0 = \omega$. □

Remark 3.58. The proof of Proposition 3.57 is based on Moser's argument on the isotopy of symplectic forms. Roughly speaking, the argument shows that for every family of symplectic forms ω^t on a manifold M with an exact time derivative $\frac{\mathrm{d}}{\mathrm{d}t}\omega^t = \mathrm{d}\alpha^t$ there exists a smooth isotopy (ψ_t) such that $\psi_t^*\omega^t = \omega^0$. The idea is to obtain this isotopy as the flow of a time–dependent vector field. Hence, suppose $\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = X_t \circ \psi_t$ and $\psi_0 = \mathrm{id}$. Then the equation $\psi_t^*\omega^t = \omega^0$ holds, provided that $\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^*\omega_t) = 0$ for all t. We have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\psi_t^*\omega_t) = \psi_t^*(\tfrac{\mathrm{d}}{\mathrm{d}t}\omega^t + \iota_{X_t}\,\mathrm{d}\omega^t + \mathrm{d}\iota_{X_t}\omega^t) = \psi_t^*(\tfrac{\mathrm{d}}{\mathrm{d}t}\omega^t + \mathrm{d}\iota_{X_t}\omega^t).$$

The relation $\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^*\omega_t) = 0$ is satisfied if $\alpha^t + \iota_{X_t}\omega_t = 0$. Since ω^t is non–degenerate there is a time–dependent vector field X_t satisfying this equation. Its flow satisfies $\psi_t^*\omega^t = \omega^0$. This argument works as outlined if M is closed. In general one has to take care of the domain of (φ_t).

Proof of Proposition 3.57. There exists a neighbourhood $\widetilde{U}_0$ of N and a 1–form $\alpha \in \Omega^1(\widetilde{U}_0)$ such that $\alpha_x = 0$ for all $x \in N$ and $\mathrm{d}\alpha = \omega^1 - \omega^0$. We postpone the proof of this claim. For $t \in [0,1]$ we define the 2–form $\omega^t = \omega^0 + t(\omega^1 - \omega^0)$. Our hypothesis that $\omega_x^1 = \omega_x^0$ for $x \in N$ implies that there exists an open neighbourhood $\widehat{U}_0 \subset \widetilde{U}_0$ of N such that $\omega^t|_{\widehat{U}_0}$ is non–degenerate for every t. We define X^t to be the unique vector field on $\widehat{U}_0$ satisfying $\alpha = -\iota_{X_t}\omega^t = -\omega^t(X_t, \_\_)$. The vector field X^t vanishes along N for all t.

It follows that there is an open neighbourhood $U_0 \subset \widehat{U}_0$ of N such that the domain of the flow (φ_t) of $X = (X_t)$ contains $[0,1] \times U_0$. We have

$$\tfrac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^* \omega^t) = \varphi_t^*(\mathcal{L}_{X_t}\omega^t + \tfrac{\mathrm{d}}{\mathrm{d}t}\omega^t) = \varphi_t^*(-\,\mathrm{d}\alpha + \omega^1 - \omega^0) = 0.$$

We define $\psi = \varphi_1|_{U_0}$ and $U_1 = \psi(U_0)$. It follows that $\psi^*\omega^1 = \psi_0^*\omega^0|_{U_0} = \omega^0|_{U_0}$. Since X^t vanishes along N for every t, its flow restricts to the identity on N. It follows that U_1 is an open neighbourhood of N.

To prove the claim, choose a tubular neighbourhood of N, i.e. an open neighbourhood $\widetilde{U}_0$ of N and a diffeomorphism $\chi\colon \nu_N \coloneqq (TM|_N)/TN \to \widetilde{U}_0$. We may e.g. choose a Riemannian metric g on M. We denote by $TN^\perp = \coprod_{x\in N} T_xN^\perp$ the normal bundle of N with repsect to g. The restriction of the exponential map of g to a neighbourhood V of the zero–section in $TN^\perp$ is an embedding. Composing this embedding with a fibre–preserving diffeomorphism between $TN^\perp$ and V and with the canonical isomorphism $\nu_N \cong TN^\perp$ we obtain a tubular neighbourhood. We define $\tau = \omega^1 - \omega^0$ and $\varphi_t\colon \widetilde{U}_0 \to \widetilde{U}_0$ with $\varphi_t(\chi(x,v)) = \chi(x,tv)$. For $t \in [0,1]$ we define $\alpha^t \in \Omega^1(\widetilde{U}_0)$ by $\alpha^t_x(v) = \tau_{\varphi_t(x)}(\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x), \mathrm{d}\varphi_t(x)v)$ and $\alpha = \int_0^1 \alpha^t\,\mathrm{d}t$. Since $\varphi_t|_N = \mathrm{id}$, α^t vanishes on N. Let $t > 0$. Then φ_t is a diffeomorphism. We may define the vector field $X_t(\tfrac{\mathrm{d}}{\mathrm{d}t}\varphi_t) \circ \varphi_t^{-1}$. We have $\tfrac{\mathrm{d}}{\mathrm{d}t}(\varphi_t^* v) = \varphi_t^*\mathcal{L}_{X_t}\tau = \mathrm{d}\varphi_t^*\iota_{X_t}\tau = \mathrm{d}\alpha^t$. Also, $\varphi_0(\widetilde{U}_0) \subset N$ and $\varphi_1 = \mathrm{id}$ and therefore $\varphi_0^*\tau = 0$ and $\varphi_1^*\tau = \tau$. We obtain $\tau = \varphi_1^*\tau - \varphi_0^*\tau = \int_0^1 \mathrm{d}\alpha^t\,\mathrm{d}t = \mathrm{d}\alpha$. □

Remark 3.59. The vector field X_t as in the proof becomes singular at $t = 0$. This does not affect the argument.

Moser's argument has another striking consequence, namely, that two symplectic forms on a closed manifold are isomorphic if they can be joined through a family of cohomologous symplectic forms:

Theorem 3.60 (Moser stability)**.** *Let M be a closed manifold. Suppose $(\omega_t)_{t\in[0,1]}$ is a smooth family of cohomologous symplectic forms on M. Then there exists a smooth isotopy $\psi = (\psi_t)$ such that $\psi_0 = \mathrm{id}$ and $\psi_t^*\omega_t = \omega_0$.*

Definition 3.61 (Isotopic symplectic forms)**.** Two symplectic forms ω_0 and ω_1 on a manifold M are called *isotopic* if they can be joined by a smooth family of cohomologous symplectic forms. The two forms are called *strongly isotopic* if there exists a smooth isotopy (ψ_t) of M such that $\psi_1^*\omega_1 = \omega_0$.

Remark 3.62. Strong isotopy implies isotopy. Theorem 3.60 shows that the converse is also true on closed manifolds. It is wrong if M is open, i.e. without boundary and with no compact connected component.

Example 3.63. Let $M = \mathbb{R}^2$ with the standard symplectic form ω_0 and $\omega_1 \coloneqq \varphi^*\omega_0$ where $\varphi\colon \mathbb{R}^2 \to B_1^2 \subset \mathbb{R}^2$ is an orientation preserving diffeomorphism. Then ω_0 and ω_1 are isotopic but not strongly isotopic, which can be shown by looking at volumes.

Corollary 3.64 (Classification of closed symplectic surfaces)**.** *Let Σ be a closed connected oriented real surface.*

1. *There exists a symplectic form on Σ.*

2. *Two symplectic forms ω_0 and ω_1 on Σ are isomorphic if and only if they have the same total area*
$$\mathrm{Vol}(\Sigma, \omega_0) = \int_\Sigma \omega_0 = \int_\Sigma \omega_1 = \mathrm{Vol}(\Sigma, \omega_1)$$

Proof.

1. Σ can be embedded in $\mathbb{R}^3$ and every oriented surface in $\mathbb{R}^3$ carries an induced symplectic form.

2. Assume $\mathrm{Vol}(\Sigma, \omega_0) = \mathrm{Vol}(\Sigma, \omega_1)$. Since Σ is connected it follows that $\omega_1 - \omega_0$ is exact. For $t \in [0,1]$ define $\omega_t = (1-t)\omega_0 + t\omega_1$. This is a symplectic form cohomologous to ω_0 and ω_1. Applying Moser stability, there is a smooth isotopy (ψ_t) of Σ such that $\psi_0 = \mathrm{id}$ and $\psi_t^* \omega_t = \omega_0$. Hence, ω_1 and ω_0 are isomorphic. □

In higher dimensions there is no such easy classification.

Example 3.65. Let (Σ, σ) and (Σ, σ') be closed connected symplectic surfaces. Write
$$c = \int_{\Sigma'} \sigma' \Big/ \int_\Sigma \sigma.$$
Let $C \in \mathbb{R} \setminus (\mathbb{Z} + c\mathbb{Z})$. The symplectic forms $\sigma \oplus \sigma'$ and $C\sigma \oplus C^{-1}\sigma'$ are symplectic forms on $\Sigma \times \Sigma'$ with the same volume which are not isomorphic. Moreover, there is a subset $X \subset (1, \infty)$ of the cardinality of the continuum such that the symplectic forms $C\sigma \oplus C^{-1}\sigma'$ with $C \in X$ are pairwise non–isomorphic.

Remark 3.66. There is a cohomological obstruction for two symplectic forms to be isomorphic: If ω and ω' are isomorphic symplectic forms on a manifold M, then there is an isomorphism $\Phi\colon H^*_{\mathrm{dR}}(X) \to H^*_{\mathrm{dR}}(X)$ that maps the integer lattice to itself and satisfies $\Phi[\omega] = [\omega']$. This lattice is the image of
$$H^*(X, \mathbb{Z}) \to H^*(X, \mathbb{R}) \to H^*_{\mathrm{dR}}(X).$$

Lemma 3.67. *Let M be a closed manifold, $k \in \mathbb{N}$ and $(\omega_t)_{t\in[0,1]}$ a smooth family of cohomologous k–forms. Then there exists a smooth family of $(k-1)$–forms α_t such that $\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \mathrm{d}\alpha_t$.*

Remark 3.68. The hypothesis of this lemma implies that for all t there exists $\widetilde{\alpha}_t$ such that $\mathrm{d}\widetilde{\alpha}_t = \omega_t - \omega_0$. If $(\widetilde{\alpha}_t)$ is a smooth family, then we are done.

Proof of Theorem 3.60. This is a consequence of Lemma 3.67 and Remark 3.58. □

Proof of Lemma 3.67. Assume first that $M = \mathbb{R}^n$ and that there exists a compact subset $K \subset M$ such that $\operatorname{supp} \omega_t \subset K$ for all t. We denote by $\pi\colon \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}^{n-1}$ and $\pi'\colon \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \to \mathbb{R}$ the canonical projection and by $\Omega^*_c(\mathbb{R}^n)$ the vector space of differential forms on $\mathbb{R}^n$ with compact support (of any degree). We define $\pi_*\colon \Omega^*_c(\mathbb{R}^n) \to \Omega^{*-1}_c(\mathbb{R}^{n-1})$ to be the unique linear map satisfying

$$(\pi^*\tau)f \mapsto 0 \qquad (\pi^*\tau)\wedge f\,\mathrm{d}x^n \mapsto \tau\int_{-\infty}^{\infty} f(\_, x^n)\;\mathrm{d}x^n, \qquad \forall \tau \in \Omega^*(\mathbb{R}^{n-1}), f \in C^\infty_c(\mathbb{R}^n),$$

which is called integration along the fiber. This map is well–defined, i.e. the image of each form $\pi^*\tau \wedge f\,\mathrm{d}x^n$ indeed has compact support. We choose a smooth function $e\colon \mathbb{R} \to \mathbb{R}$ with compact support and integral 1 and define

$$e_*\colon \Omega^*_c(\mathbb{R}^{n-1}) \to \Omega^{*+1}_c(\mathbb{R}^n), \qquad \tau \mapsto (\pi^*\tau)\wedge(e\circ\pi')\,\mathrm{d}x^n$$

We also define $E\colon \mathbb{R} \to \mathbb{R}$ by $E(s) = \int_{-\infty}^{s} e(s')\;\mathrm{d}s'$, and $H\colon \Omega^*_c(\mathbb{R}^n) \to \Omega^{*-1}_c$ to be the unique linear map satisfying, for all $\tau \in \Omega^*(\mathbb{R}^{n-1})$ and $f \in C^\infty(\mathbb{R}^n)$

$$(\pi^*\tau)f \mapsto 0$$
$$(\pi^*\tau)\wedge f\,\mathrm{d}x^n \mapsto (\pi^*\tau)\left(\int_{-\infty}^{\pi'(\_)} f(\pi(\_), x^n)\;\mathrm{d}x^n - E\circ\pi'\int_{-\infty}^{\infty} f(\pi(\_), x^n)\;\mathrm{d}x^n\right)$$

We claim that

$$(1 - e_*\pi_*)\tau = (-1)^{k-1}(\,\mathrm{d}H - H\,\mathrm{d})\tau \qquad \forall \tau \in \Omega^k_c(\mathbb{R}^n).$$

This means that H is a chain homotopy between the identity and $e_*\pi_*$. This is proved in Assigment 10 and can be found in Bott and Tu, Differential forms in Algebraic Topology, rev. 3, Spring, 1982, Prop. 4.6, p.38.

Let $\tau \in \Omega^k(\mathbb{R}^n)$ be exact, and therefore closed. Then by the above formula and $\pi_*\,\mathrm{d} = \mathrm{d}\pi_*$, we get

$$(\pi_*)^i\tau = e_*(\pi_*)i + 1\tau + \,\mathrm{d}H(\pi_*)^i\tau \qquad \forall i < k.$$

Also we have, with $\tau = \,\mathrm{d}\alpha$,

$$(\pi_*)^k = (\pi_*)^k\,\mathrm{d}\alpha = \pi_*\,\mathrm{d}(\pi_*)^{k-1}\alpha = 0,$$

since $\pi_*\,\mathrm{d}f = 0$ for a function f. Successively inserting these formulas into the next, we get

$$\tau = \,\mathrm{d}H\tau + e_*\pi_*\tau = \,\mathrm{d}H\tau + \,\mathrm{d}e_*H\pi_*\tau + e_*e_*\pi_*\pi_*\tau = \cdots = \,\mathrm{d}\left(\sum_{i=0}^{k-1}(e_*)^i H(\pi_*)^i\right)\tau$$

Let now $(\omega_t)_{t\in[0,1]}$ be as in the hypothesis of the lemma and $t \in [0,1]$. Then $\tau_t = \omega_t - \omega_0$ is exact and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}\omega_t = \frac{\mathrm{d}}{\mathrm{d}t}\tau_t = \,\mathrm{d}\left(\sum_{i=0}^{k-1}(e_*)^i H(\pi_*)^i\right)\tau_t.$$

The statement of the claim in the case $M = \mathbb{R}^n$ follows. In general, the statement follows from this and an induction argument over the minimal number of parametrizations $\psi_i\colon \mathbb{R}^n \to M$ needed to cover M.

□

Remark 3.69. Compare this to the proof of the Poincaré Lemma with compact support.

3.5 Symplectic, (co-)isotropic and Lagrangian submanifolds

Let (M, ω) be a symplectic manifold and $N \subset M$ an embedded submanifold.

Definition 3.70. N is called *symplectic/isotropic/coisotropic/Lagrangian* if for every $x \in N$ the subspace $T_x N$ of $(T_x M, \omega_x)$ is symplectic/isotropic/coisotropic/Lagrangian.

Example 3.71. If (M, ω) and (M', ω') are symplectic manifolds and $x' \in M'$ then $M \times \{x'\}$ is a symplectic submanifold of $(M \times M', \omega \oplus \omega')$. Let X be a manifold. For $q \in X$ the fiber $T_q^* X$ is Lagrangian: Let $p \in T_q^* X$, $x = (q, p)$ and $u \in T_x(T_q^* X) = T_q^* X$. By the definition of λ^{can} we have

$$\lambda_x^{\mathrm{can}} v = p \, \mathrm{d}\pi(x) v = 0.$$

Hence the pullback of λ^{can} under the inclusion $T_q^* X \to T^* X$ vanishes. Thus the same holds for the pullback of $\omega^{\mathrm{can}} = -\mathrm{d}\lambda^{\mathrm{can}}$. This shows that $T_q^* X$ is Lagrangian with respect to ω^{can}.

Let (M, ω) be a symplectic manifold. Every one–dimensional submanifold of M is isotropic. Every hypersurface in M is coisotropic. More examples of Lagrangian submanifolds:

Proposition 3.72. *Let (M, ω) be a symplectic manifold and $\varphi \colon M \to M$ a diffeomorphism. Then φ is a symplectomorphism if and only if its graph is a Lagrangian submanifold with respect to $(-\omega) \oplus \omega$.*

Proof. Assignment 10.

Corollary 3.73. *The diagonal $\{(x, x) \mid x \in M\} \subset M \times M$ is Lagrangian with respect to $(-\omega) \oplus \omega$.*

Proposition 3.74. *Let X be a manifold and $\alpha \in \Omega^1(X)$. Then the graph of α is a Lagrangian submanifold of $T^* X$ if and only if α is closed.*

Proof. The form ω^{can} vanishes on the graph of α if and only if $\alpha^* \omega^{\mathrm{can}} = 0$. By Proposition 3.12 $\alpha^* \lambda^{\mathrm{can}} = \alpha$, and therefore $\alpha^* \omega^{\mathrm{can}} = \alpha^*(-\mathrm{d}\lambda^{\mathrm{can}}) = -\mathrm{d}\alpha$. It follows that ω^{can} vanishes on $\mathrm{gr}\, \alpha$ if and only if $\mathrm{d}\alpha = 0$. ☐

Corollary 3.75. *The zero section in $T^* X$ is Lagrangian.*

Proof. This follows from Proposition 3.74 with $\alpha = 0$. ☐

Exercise 3.76. Let X be a manifold and $Y \subset X$ a submanifold. Prove that the annihilitor

$$TY^0 := \{(q, p) \in T^* X \mid q \in Y, p|_{T_q Y} = 0\}$$

is a Lagrangian submanifold of $T^* X$ with respect to ω^{can}.

Proposition 3.72, Proposition 3.74 and Exercise 3.76 show that Lagrangian submanifolds show up in various situations. In fact, the symplectic geometer A. Weinstein came up with the motto 'everything is a Lagrangian submanifold' and he created a dictionary in which he interpreted many concepts of symplectic geometry in terms of Lagrangian submanifolds.

Example 3.77. Let (M, ω) be a symplectic manifold and $\varphi \in \operatorname{Symp}(M, \omega)$. The fixed points of φ correspond bijectively to intersection points of the graph of φ with the diagonal $\Delta := \{(x, x) \mid x \in M\} \subset M \times M$ via the map $x \mapsto (x, x)$. By Proposition 3.72 $\operatorname{gr}\varphi$ and Δ are Lagrangian submanifolds of $(M \times M, (-\omega) \oplus \omega)$.

Symplectic, isotropic, coisotropic and Lagrangian submanifolds admit normal forms. These are consequences of a normal form for a general submanifold. In order to state this result we need:

Definition 3.78. Let X be a smooth manifold and $\pi\colon E \to X$ a real (smooth) vector bundle. A *symplectic bilinear form* on E is a collection $\omega := (\omega_x)_{x \in X}$, where ω_x is a symplectic bilinear form on E_x which varies smoothly in x. We call (E, π, ω) a *(smooth) symplectic vector bundle.* An isomorphism (covering the identity on the base) between two symplectic vector bundles (E, π, ω) and (E', π', ω') over X is a (smooth) isomorphism of vector bundles $\Phi\colon E \to E'$ such that $\pi' \circ \Phi = \pi$ and $\Phi^*\omega' = \omega$.

Remark 3.79. The smoothness condition on ω means that ω is a smooth section of the exterior product $E^* \wedge E^*$.

Example 3.80. Let $E \to X$ be a smooth vector bundle. We equip $E \oplus E^*$ with the symplectic bilinear form ω_E defined by

$$\omega_E((w, \varphi), (w', \varphi')) := \varphi'(w) - \varphi(w').$$

The pair $(E \oplus E^*, \omega_E)$ is a symplectic vector bundle, which generalizes the definition of the canonical bilinear form on $W \times W^*$ for a vector space W. Let (E, ω) be a symplectic vector bundle over X and $W \subset E$ a subbundle. We denote

$$W^\omega = \{(x, v) \mid x \in X, v \in W_x^{\omega_x}\}$$

This is a (smooth) subbundle of E. Assume that $W \cap W^\omega$ has constant rank. Then this is a subbundle of E. We endow the quotient bundle $W/(W \cap W^\omega)$ with the bilinear symplectic form ω^W defined by

$$\omega^W(v + W \cap W^\omega, v' + W \cap W^\omega) := \omega(v, v')$$

Exercise 3.81. Prove that this is a symplectic bilinear form. Compare to subsection 2.3.

The following theorem is the main result of this section:

Theorem 3.82 (Normal form)**.** *Let M be a manifold, ω_0 and ω_1 symplectic forms on M and $N \subset M$ a closed submanifold. We denote by $\iota\colon N \to M$ the inclusion. Assume that $\iota^*\omega_0 = \iota^*\omega_1$ and $K := TN \cap TN^{\omega_0}$ has constant rank and the symplectic vector bundles $(TN^{\omega_i}/K, \omega_i^{TN^{\omega_i}})$ with $i \in \{0, 1\}$ are isomorphic. Then there exist neighbourhoods U_0 and U_1 of N and a diffeomorphism $\varphi\colon U_0 \to U_1$ such that $\varphi|_N = \mathrm{id}$ and $\varphi^*\omega_1 = \omega_0$.*

Remark 3.83. We have $K = TN \cap TN^{\omega_0} = TN \cap TN^{\omega_1} = \ker(TN \ni (x,v) \mapsto (x,(\omega_i)_x(v,\_)|_{T_xN}) \in T^*N)$. Theorem 3.54 is a consequence of Theorem 3.82.

Corollary 3.84 (A Weinstein symplectic neighbourhood theorem)**.** *Assume that $\iota^*\omega_0 = \iota^*\omega_1$, this restriction is symplectic and the bundles $(TN^{\omega_i}, \omega_i|_{TN^{\omega_i}})$ are isomorphic for $i \in \{0,1\}$. Then there exist U_0, U_1 and φ as in the conclusion of Theorem 3.82.*

Corollary 3.85 (A Weinstein Lagrangian neighbourhood theorem)**.** *Let (M,ω) be a symplectic manifold and $L \subset M$ a closed Lagrangian submanifold. Then there exists an open neighbourhood $U \subset T^*L$ of the zero section and $V \subset M$ of L and a diffeomorphism $\varphi\colon U \to V$ such that $\varphi|_L = \mathrm{id}$ and $\varphi^*\omega = \omega^{\mathrm{can}}$.*

Proof. Since L is Lagrangian, the map $\Phi\colon \nu_L = (TM|_L)/TL \to T^*L$ defined by $\Phi(x, v + T_xL) \coloneqq (x, \omega_x(v,\_))$ is a well–defined isomorphism (Check this!). We choose a tubular neighbourhood for L, i.e. an open neighbourhood $\widetilde{V} \subset M$ of L and a diffeomorphism $\chi\colon \nu_L \to \widetilde{V}$ that is the identity on L. Compare to the proof of Proposition 3.57. Applying Theorem 3.82 to the forms $\omega_0 = \omega^{\mathrm{can}}$ and $\omega_1 = (\Phi^{-1})^*\chi^*\omega$ there exist open neighbourhoods U_0, U_1 of L in T^*L and a diffeomorphism $\psi\colon U_0 \to U_1$ satisfying $\psi|_L = \mathrm{id}$ and $\psi^*\omega_1 = \omega_0$. The triple $U \coloneqq U_0$, $V \coloneqq \chi \circ \Phi^{-1}(U_1)$, $\varphi \coloneqq \chi \circ \Phi^{-1} \circ \psi$ has the required properties, since $\varphi^*\omega = \psi^*(\Phi^{-1})^*\chi^*\omega = \psi^*\omega_1 = \omega_0 = \omega^{\mathrm{can}}$. □

Remark 3.86. Together with the fact that the embedding of L as the zero section of T^*L is Lagrangian, this result classifies germs of neighbourhoods of Lagrangian embeddings of a given manifold L up to diffeomorphisms of such neighbourhoods that intertwine the embeddings and the symplectic structures.

Theorem 3.82 immediately implies normal form results for (co–)isotropic submanifolds. (See Assignment 11).

The proof of Theorem 3.82 is based on Proposition 3.57, which was already used in the proof of Darboux' Theorem, Theorem 3.54. We also need

Proposition 3.87 (Whitney Extension Theorem)**.** *Let M be a manifold, $N \subset M$ a submanifold and Ψ a smooth automorphism of $TM|_N$ that restricts to the identity on TN. Then there exists an open neighbourhood $U \subset M$ of N and an embedding $\psi\colon U \to M$ such that $\psi|_N = \mathrm{id}$ and $\mathrm{d}\psi(x) = \Psi x$ for all $x \in N$.*

Remark 3.88. The automorphism Ψ in this result serves as a germ for the embedding ψ.

Proof. Assignment 11.

Proposition 3.89 (Subbundles of symplectic vector bundles)**.** *Let (E,ω) be a symplectic vector bundle over a manifold X and $W_0 \subset W \subset E$ subbundles such that $W = W_0 \oplus K$, where $K \coloneqq W \cap W^\omega$. Then there exists an iosmoprhism*

$$\Phi\colon (E,\omega) \to (W/K, \omega^W) \oplus (W^\omega/K, \omega^{W^\omega}) \oplus (K \oplus K^*, \omega_K)$$

such that $\Phi(w_0 + w) = (w_0 + K, 0, w, 0)$ for all $w_0 \in W_0$ and $w \in K$.

Proof of Theorem 3.82. Since by hypothesis $K := TN \cap TN^{\omega_0}$ has constant rank, there exists a subbundle $W_0 \subset TN$ such that $TN = W_0 \oplus K$. By Proposition 3.89 for $i \in \{0,1\}$ there exists an isomorphism

$$\Phi_i\colon (TM|_N, \omega_i) \to (TN/K, \omega_i^{TN}) \oplus (TN^{\omega_i}/K, \omega_i^{TN^{\omega_i}}) \oplus (K \oplus K^*, \omega_K)$$

such that $\Phi_i(w_0 + w) = (w_0 + K, 0, w, 0)$ for all $w \in W_0$ and $w \in K$. By hypothesis there exists an isomorphism between symplectic the vector bundles $(TN^{\omega_i}/K, \omega_i^{TN^{\omega_i}})$. Combining such an isomorphism with Φ_0 and Φ_1 we obtain an isomorphism $\Psi\colon (TM|_N, \omega_0) \to (TM|_N, \omega_1)$ that restricts to the identity on TN (Check this!). By Proposition 3.87 there exists an open neighbourhood $U \subset M$ of N and an embedding $\psi\colon U \to M$ such that $\psi|_N = \mathrm{id}$ and $\mathrm{d}\psi(x) = \Psi x$ for all $x \in N$. It follows that $\psi^*\omega_1 = \Psi^*\omega_1 = \omega_0$ along N and the triple N, ω_0, ω_1 satisfies the requirements of Proposition 3.87. Applying that result there exist neighbourhoods $U_0 \subset U$ and $U_1 \subset \psi(U)$ and a diffeomorphsim $\psi'\colon U_0 \to U_1$ that restricts to the identity on N and satisfies $\psi'^*\omega_1 = \omega_0$. The triple $U_0, U_1, \varphi := \psi \circ \psi'$ has the properties required by the conclusion of Theorem 3.82. □

We will prove Proposition 3.89 by reducing to the case in which $W \subset E$ is a Lagrangian subbundle. In this case we may extend the proof of Theorem 2.53 to the vector bundle setting. The main ingredient is

Lemma 3.90 (Lagrangian complement)**.** *Let X be a manifold, (E, ω) a symplectic vector bundle over X and $W \subset E$ a Lagrangian subbundle. Then there exists a (smooth) Lagrangian subbundle $W' \subset E$ that is complementary to W, i.e. $W \cap W' = 0$.*

Remark 3.91. This is a generalization of Lemma 2.58.

Proof of Proposition 3.89. Consider the case in which W is Lagrangian. Then $W_0 = 0$. We choose a complementary Lagrangian subbundle $W' \subset E$ as in Lemma 3.90. We define

$$\Phi\colon E \to W \times W^*, \qquad \Phi(x, v) := (x, w, \omega(w', \_)|_W)$$

where $x \in X$ and $(w, w') \subset W_x \times W'_x$ is the unique pair satisfying $v = w + w'$. It follows as in the proof of Theorem 2.53 that this map is well–defined and restricts to an isomorphism $E_x \to W_x \times W_x^*$, for every $x \in X$, and satisfies $\Phi^*\omega_W = \omega$ (Check this!). Furthermore, Φ is smooth. So it has the disired properties. This proves Proposition 3.89 if W is Lagrangian. Consider now the general situation. We choose a subbundle W_1 of W^ω that is complementary to $K := W \cap W^\omega$, i.e. $W^\omega = W_1 \oplus K$. We may for example choose a bundle metric on W^ω and define $W_1 := K^\perp$ with respect to this metric. We define $E' := (W_0 \oplus W_1)^\omega$. It follows from linear algebra that W_i is a symplectic subbundle of E for $i \in \{0,1\}$ (Check this!). Therefore, by what we proved above, there exists an isomorphism $\Phi'\colon (E', \omega|_{E'}) \to (K \oplus K^*, \omega_K)$ satisfying $\Phi'(w, 0) = (w, 0)$ for all $w \in K$. We define the map

$$\Phi\colon E \to (W/K) \oplus (W^\omega/K) \oplus K \oplus K^*, \qquad \Phi(x, v) := (x, w_0 + K, w_1 + K, \Phi' w'),$$

where $w_i \in (W_i)_x$ and $w' \in E'_x$ are the unique vectors such that $v = w_0 + w_1 + w'$. This map has the required properties. □

Proof of Lemma 3.90. The statement follows from the proof of Lemma 2.58 using the fact that there exists a smooth subbundle U inside E that is complementary to W. □

Remark 3.92. There is an alternative construction of a Lagrangian complement of a Lagrangian subspace W of (V, ω). Choose a linear structure J on V that is ω–compatible. Define $W' = JW$.

Theorem 3.93. *Let (M, ω) be a closed symplectic manifold, and equip the group of Hamiltonian diffeomorphisms $\operatorname{Ham}(M, \omega)$ with the C^1–topology. Define $\operatorname{Crit} M$ to be the minimal number of critical points of a smooth real valued function on M. There exists a C^1–neighbourhood U of the identity in $\operatorname{Ham}(M, \omega)$ such that every $\varphi \in U$ has at least $\operatorname{Crit} M$ fixed points.*

Remark 3.94. This is the statement of the Arnold Conjecture, but we have the additional C^1–closeness hypothesis.

Proof. If $H^1_{\mathrm{dR}}(M) = 0$ then this follows from Assignment 11, Exercise 5. The idea of the proof is the following: By Corollary 3.85 we find $\psi \colon U \subset M \times M \to V \subset T^*\Delta$ (where $\Delta = \{(x, x) \mid x \in M\}$ is the diagonal submanifold) such that $\psi^*\omega = \omega^{\mathrm{can}}$ and $\psi|_\Delta = \mathrm{id}$. If $\varphi \in \operatorname{Ham}(M, \omega)$ is C^1–close to the identity, then $\operatorname{gr}\varphi \subset U$ and $\psi(\operatorname{gr}\varphi)$ is C^1–close to the zero section in $T^*\Delta \cong T^*M$, so it is the graph of a 1–form $\alpha \in \Omega^1(M)$. By Proposition 3.72, $\operatorname{gr}(\varphi)$ is Lagrangian and by therefore, by Proposition 3.74, $\mathrm{d}\alpha = 0$. Since $H^1_{\mathrm{dR}}(M) = 0$ by assumption, α is exact, i.e there exists a smooth function $f \colon M \to \mathbb{R}$ such that $\mathrm{d}f = \alpha$. So the fixpoints of φ correspond to critical points of f. The statement of the theorem follows. For the general situation without $H^1_{\mathrm{dR}}(M) = 0$ see [MS:344] IST Proposition 11.5. □

3.6 Symplectic and Hamiltonian Lie group actions, momentum maps, Marsden–Weinstein quotients

Noether's Theorem states that every continous symmetry of configuration space that preservces the Lagrangian function gives rise to a conserved quantity (integral of motion). Such a symmetry gives rise to a Hamiltonian action of the Lie group $\mathbb{R}$ on phase space T^*X that preserves the Hamiltonian function corresponding to the Lagrangian function via Legendre transform.

Conserved quantities are important in mechanics, since they can be used to decrease the number of degrees of freedum. The notion of a symplectic (ore Marsden–Weinstein) quotient makes this precise. Such quotients are associated with Hamiltonian Lie group actions. the definition of such an action involves the notion of momentum maps, which generalizes linear and angular momentum. We will see how to generalize Marsden–Weinstein quotients to symplectic quotients of regular presymplectic manifolds and hence of regular coisotropic submanifolds. This yields many examples of symplectic manifolds.

Let (M, ω) be a symplectic manifold, G a Lie group and $\mathfrak{g} = \operatorname{Lie} G = T_1 G$ the Lie algebra of G.

Definition 3.95. The *adjoint representation* (or action) of G is

$$\mathrm{Ad} = \mathrm{Ad}_G \colon G \times \mathfrak{g} \to \mathfrak{g}, \qquad \mathrm{Ad}_g \xi \coloneqq \mathrm{Ad}(g, \xi) \coloneqq \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} g \exp(t\xi) g^{-1} \qquad \forall g \in G$$

where $\exp\colon \mathfrak{g} \to G$ denotes the exponential map, i.e. $\exp(\xi) = \gamma(1)$ if $\gamma\colon \mathbb{R} \to G$ is the unique smooth group homomorphism with $\dot{\gamma}(0) = \xi$.

Definition 3.96. A (smooth) action of G on M is called *symplectic* if every element of G acts by a symplectomorphism. It is called *Hamiltonian* if it is symplectic and there exists a smooth function $\mu\colon M \to \mathfrak{g}^*$ such that $\mu(gx) = \mathrm{Ad}^*_{g^{-1}} \mu(x) = \mu(x) \mathrm{Ad}_{g^{-1}}$ and $\mathrm{d}\langle \mu, \xi\rangle = \omega(X_\xi, \_)$. A map with these properties is called *momentum map* for the action of G on M. Here $\mathfrak{g}^*$ is the dual of $\mathfrak{g}$, $\langle \mu, \xi \rangle \colon M \to \mathbb{R}$ is defined by $x \mapsto \langle \mu(x), \xi\rangle = \mu(x)\xi$ and $(X_\xi)_x = D_e R_x(\xi)$ (where $R_x\colon G \to M, g \mapsto gx$) is the fundamental vector field of ξ.

Remark 3.97. This means that μ is equivariant with respect to $G \circlearrowright M$ and the coadjoint representation of G (the dual of the adjoint representation). The condition $\mathrm{d}\langle \mu, \xi\rangle = \omega(X_\xi, \_)$ means that μ generates the infinitesimal action of g on M.

A momentum map is in general not unique. However if M is connected then it is unique up to adding a central element $\varphi \in \mathfrak{g}^*$. This means that $\mathrm{Ad}^*_g \varphi = \varphi$ for all $g \in G$.

The next exercise shows that if G is connected then in the definition of a Hamiltonian action the condition that the action is symplectic is redundant.

Exercise 3.98. Let (M, ω) be a symplectic manifold and G a Lie group. Fix a smooth action of G on M for which there exists a smooth map $\mu\colon M \to \mathfrak{g}$ satisfying $\mathrm{d}\langle \mu, \xi\rangle = \omega(X_\xi, \_)$ for all $x \in M$ and $\xi \in g$. If G is connected then every element of G acts by a Hamiltonian diffeomorphism.

Remark 3.99. It follows that the action is symplectic.

Example 3.100. The trivial action of a Lie group on a symplectic manifold is Hamiltonian. Its momentum map is $\mu = 0\colon M \to \mathfrak{g}^*$.

Example 3.101 (standard Hamiltonian action of the unitary group)**.** Let $k, n \in \mathbb{N}$ and $G \coloneqq U(k)$ act on $\mathbb{C}^{k\times n}$ by left multipliction of matrices. We equip $\mathbb{C}^{k\times n} = \mathbb{R}^{2kn}$ with the standard symplectic form ω_0. We define the inner product $\langle \_, \_\rangle$ on $\mathfrak{u}(k) \coloneqq \mathrm{Lie}\, U(k)$ by $\langle \zeta, \zeta'\rangle = \mathrm{tr}(\zeta^*\zeta')$. The map

$$\mu\colon \mathbb{C}^{k\times n} \to \mathfrak{u}(k)^*, \qquad \langle \mu(\Theta), \xi\rangle \coloneqq \langle \tfrac{i}{2}(1 - \Theta\Theta^*), \xi\rangle$$

is a momentum map for the action (Assignment 12).

Remark 3.102. If $k = 1$ in this example then the circle $U(1) = S^1$ acts on $\mathbb{C}^{1\times n} = \mathbb{C}^n$ via $w \cdot z \coloneqq (wz_1, \ldots, wz_n)$, with the moment map $\mu\colon \mathbb{C}^n \to \mathrm{Lie}\, S^1 = i\mathbb{R}$ given by $\mu(z) :- \frac{i}{2}(1 - |z|^2)$. Here we identify $\mathrm{Lie}\, S^1$ with its dual via the standard inner product $\langle \_, \_\rangle$ on $\mathrm{Lie}\, S^1$. Under the canonical identification

$$\mathbb{R}/2\pi\mathbb{Z} \ni (t + 2\pi\mathbb{Z}) \mapsto e^{it} \in S^1$$

this momentum map corresponds to the function $H\colon \mathbb{C}^n \to \mathbb{R}$ given by $H(z) \coloneqq \frac{1}{2}(1 - |z|^2)$. Threfore, it follows from Assignment 1, Exercise 5 and Remark 3.103 below that $\frac{i}{2}(1 - |z|^2)$ is indeed a momentum map of the action of S^1 on $\mathbb{C}^n$.

Remark 3.103 (Hamiltonian actions of the real line). Let (M, ω) be a symplectic manifold. A smooth action of $\mathbb{R}$ on M is Hamiltonian if and only if X_1, the fundamental vector field of $1 \in \operatorname{Lie}\mathbb{R} = \mathbb{R}$, is Hamiltonian. In this case the map $\mu\colon M \to \mathbb{R}^*$ satisfying $\langle \mu(x), \xi\rangle := H(x)\xi$, is a momentum map. Here $H\colon M \to \mathbb{R}$ is any function with Hamiltonian vector field X_1.

Example 3.104. We define $M := S^2 \subset \mathbb{R}^3$ and equip it with the standard symplectic form (see Assignment 1, Exercise 6). The circle action on S^2 given by rotation about the x_3–axis is Hamiltonian with the Hamiltonian function $H(x) = x_3$ given by the height function. This follows from Assignment 9, Exercise 2.

Remark 3.105 (action induced by a Lie group homomorphism). Let (M, ω) be a symplectic manifold, G and G' Lie groups and $\varphi\colon G' \to G$ a Lie group homomorphism. We fix a Hamiltonian action of G on M with momentump map $\mu\colon M \to \mathfrak{g}^*$. Then the map

$$G' \times M \to M, \qquad (g', x) \mapsto \varphi(g')x$$

is a Hamiltonian action of G' with moment map $\mu = \mu\, \mathrm{d}\varphi(1)\colon M \to \mathfrak{g}'^* = (\operatorname{Lie} G')^*$ (see Assignment 12).

In particular, using Remark 3.103 if $S^1 \cong \mathbb{R}/\mathbb{Z}$ acts on (M, ω) in a Hamiltonian way with Hamiltonian function (corresponding to the momentum map) $H\colon M \to \mathbb{R}$ then $\mathbb{R}$ acts on M in a Hamiltonian way via $t \cdot x := (t + \mathbb{Z})x$ with the same H. Question: Is the converse also true? Given a Hamiltonian action of $\mathbb{R}$, do we get a Hamiltonian circle action?

Example 3.106. Let G be a Lie group. For $g \in G$ we denote by

$$L_g, R_g\colon G \to G, \qquad L_g h = gh, \qquad R_g h = hg$$

the left and right translations by g. The action of G on T^*G given by

$$g \cdot (h, \varphi) := (gh, \varphi\, \mathrm{d}L_g(h)^{-1})$$

is Hamiltonian with moment map

$$\mu\colon T^*G \to \mathfrak{g}^*, \qquad \mu(g, \varphi) := \varphi\, \mathrm{d}R_g(1).$$

See Assignment 12.

Remark 3.107. The condition that μ is G–equivariant can not be dropped from the definition of a Hamiltonian action. Consider e.g. $M = \mathbb{R}^2$ equipped with the standard symplectic form and the standard action of $\mathbb{R}^2$ on $\mathbb{R}^2$ given by addition. The map

$$u\colon \mathbb{R}^2 \to (\operatorname{Lie}\mathbb{R}^2)^* = \mathbb{R}^2, \qquad \langle \mu(q,p), (\xi,\eta)\rangle := p\xi - q\eta$$

generates the infinitesimal action of $\operatorname{Lie}\mathbb{R}^2$ on $\mathbb{R}^2$ (Check this!). However, the action is not Hamiltonian (i.e. the map μ cannot by chosen invariant under the action). See Assignment 12.

Remark 3.108. The $GL(k,\mathbb{C})$–action on $\mathbb{C}^{k\times n}$ is not Hamiltonian. In fact, it is not even symplectic, e.g. $\frac{1}{2}$ does not preserve ω_0.

Given a Hamiltonian action, we can obtain a Hamiltonian S^1–action out of it if and only if $\mathbb{Z}$ acts trivially. If this is the case, for any $t+\mathbb{Z}\in\mathbb{R}/\mathbb{Z}\cong S^1$, define $(t+\mathbb{Z})x := t\cdot x$.

Example 3.109 (exact actions are Hamiltonian, induced action on a cotangent bundle)**.** Let M be a manifold, λ a 1–form on M, G a Lie group and $\varphi\colon G\times M\to M$ an action. We denote by $\xi_M = X_\xi$ the infinitesimal action of $\xi\in\mathfrak{g}$ on M. We define

$$\omega := -\,\mathrm{d}\lambda, \qquad \mu\colon M\to\mathfrak{g}^*, \quad \langle\mu,\xi\rangle = \iota_{\xi_M}\lambda = \lambda(\xi_M).$$

Assume that φ preserves λ, i.e. $\varphi_g^*\lambda = \lambda$ for all $g\in G$. Then μ is a momentum map for the pair (ω,φ). Hence the action φ is Hamiltonian (See Assignment 12). In particular, let X be a manifold. We fix a smooth action of G on X. We define the induced action φ of G on T^*X by

$$\varphi_g(q,p) := (gq, p\,\mathrm{d}L_g(q)^{-1}) \in T^*X$$

By Proposition 3.7 the map $\varphi_g\colon T^*X\to T^*X$ preserves the canonical 1–form λ^{can}. It follows that the action φ is hamiltonian with respect to $\omega^{\mathrm{can}} = -\,\mathrm{d}\lambda^{\mathrm{can}}$ with the momentum map $\mu\colon T^*X\to\mathfrak{g}^*$ given by

$$\langle\mu(x),\xi\rangle := \lambda^{\mathrm{can}}\xi_{T^*X}(x) = p\,\mathrm{d}\pi(x)\xi_{T^*X} = p\xi_X(q)$$

for every $x=(q,p)\in T^*X$ and $\xi\in\mathfrak{g}$.

Remark 3.110. The statement of Example 3.106 follows from Example 3.109. The action of G on T^*G induced by left multiplication of G on G is Hamiltonian with momentum map $\langle\mu(g,\varphi),\xi\rangle = \varphi\,\mathrm{d}R_g(1)$ (Check this!).

Remark 3.111. An action as in Example 3.109 is called exact (with respect to λ).

The next result characterizes Hamiltonian actions of connected Lie groups.

Proposition 3.112. *Let (M,ω) be a symplectic manifold and G a Lie group. We fix an action of G on M. If G is connected then the following holds:*

1. *If μ is a momentum map for the action, then the map*

$$\mathfrak{g}\to C^\infty(M), \qquad \xi\mapsto\langle\mu,\xi\rangle$$

is a homomorphism of Lie algebras with respect to the Lie bracket on $\mathfrak{g}$ and the Poisson bracket on $C^\infty(M)$.

2. *If there exists a homomorphism of Lie algebras*

$$H\colon\mathfrak{g}\to C^\infty(M), \qquad \xi\mapsto H_\xi$$

such that the fundamental vector field X_ξ equals the Hamiltonian vector field X_{H_ξ}, for every $\xi\in\mathfrak{g}$, then the action is Hamiltonian with momentum map

$$\mu\colon M\to\mathfrak{g}^*, \qquad \langle\mu(x),\xi\rangle = H_\xi(x).$$

The proof of the second part of this result is based on

Lemma 3.113. *Let M be a manifold, G a Lie group, and $\varphi\colon G \times M \to M$ a smooth action. Then we have $X_{\mathrm{Ad}_{g^{-1}}\xi} = \varphi_g^* X_\xi$ for all $g \in G$ and $\xi \in \mathfrak{g}$.*

Proof. Assignment 12.

Proof of Proposition 3.112 part two. We choose a map $H\colon \mathfrak{g} \to C^\infty(M)$ as in the hypothesis. We define $\mu\colon M \to \mathfrak{g}^*$ by $\langle \mu(x), \xi\rangle := H_\xi(x)$ for all $x \in M$ and $\xi \in \mathfrak{g}$. We show that this is a momentum map: We have $\mathrm{d}\langle \mu, \xi\rangle(x) = \mathrm{d}H_\xi(x) = \omega(X_\xi(x), \_)$ for all $x \in M$ and $\xi \in \mathfrak{g}$. Therefore, μ generates the infinitesimal action. We show that μ is G–equivariant: Let $x \in M$, $g \in G$ and $\xi \in \mathfrak{g}$. We claim that $\langle \mu(gx), \xi\rangle = \langle \mu(x), \mathrm{Ad}_{g^{-1}} \xi\rangle$. To show this, observe that by Exercise 3.98 G acts by Hamiltonian diffeomorphisms. Combining this with Lemma 3.113 and the fact that $\mathrm{d}H_\eta = \omega(X_\eta, \_)$, it follows that

$$\mathrm{d}H_{\mathrm{Ad}_{g^{-1}}\xi} = \omega(X_{\mathrm{Ad}_{g^{-1}}\xi}, \_) = \omega(\varphi_g^* X_\xi, \_) = \omega(X_\xi, (\varphi_g^{-1})^* \_) = \varphi_g^* \,\mathrm{d}H_\xi = \mathrm{d}(H_\xi \circ \varphi_g).$$

It follows that $H_{\mathrm{Ad}_{g^{-1}}\xi} - H_\xi \circ \varphi_g$ is constant. (We assume that M is connected). Therefore, we have

$$H_{[Ad_{g^{-1}}\xi, \mathrm{Ad}_{g^{-1}}\eta]} = \{H_{\mathrm{Ad}_{g^{-1}}\xi}, H_{\mathrm{Ad}_{g^{-1}}\eta}\} = \{H_\xi \circ \varphi_g, H_\eta \circ \varphi_g\}$$

for every $\eta \in \mathfrak{g}$. Since by assumption, G is connected, there exists a smooth path $h\colon [0,1] \to G$ satisfying $h(0) = 1$ and $h(1) = g$. For $g_0, g_1 \in G$ and a vector $v \in T_{g_0}G$ we denote by $vg_1 = \mathrm{d}_{g_0} R_{g_1}(v)$ the differential of the right translation $R_{g_1}\colon G \to G$ at g_0 applied to v. We define the path $\eta := \dot{h}h^{-1}\colon [0,1] \to \mathfrak{g}$. Then we have $\frac{\mathrm{d}}{\mathrm{d}t}\varphi_h = X_\eta \circ \varphi_h$ ($\frac{\mathrm{d}}{\mathrm{d}t}(h(t)x) = (\dot{h}(t)h(t)^{-1})h(t)x$), and therefore

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}(H_\xi \circ \varphi_h) &= \mathrm{d}(H_\xi \circ \varphi_h)\varphi_h^* X_\eta = \omega(X_{H_\xi \circ \varphi_h}, \varphi_h^* X_\eta) \\ &= \omega(X_{H_\xi \circ \varphi_h}, X_{H_\eta \circ \varphi_h}) = \{H_\xi \circ \varphi_h, H_\eta \circ \varphi_h\} \qquad (**) \end{aligned}$$

using Proposition 3.48. We have $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{Ad}_{g^{-1}}\xi) = \mathrm{Ad}_{h^{-1}}[\xi, \eta]$ and therefore $\frac{\mathrm{d}}{\mathrm{d}t} H_{\mathrm{Ad}_{h^{-1}}\xi} = H_{\mathrm{Ad}_{h^{-1}}[\xi,\eta]} = \{H_\xi \circ \varphi_h, H_\eta \circ \varphi_h\}$, where we used that $\eta \mapsto H_\eta(x)$ is linear. Combining this with $(**)$, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}(H_\xi \circ \varphi_h - H_{\mathrm{Ad}_{h^{--1}}\xi}) = 0$. Integrating this equality in t andusing that $\varphi_{h(0)} = 1 = \mathrm{id}$, we obtain $H_\xi \circ \varphi_g = H_{\mathrm{Ad}_{g^{-1}}\xi}$. The equality $\langle \mu(gx), \xi\rangle = \langle \mu(x), \mathrm{Ad}_{g^{-1}}\xi\rangle$ follows. This proves the claim and completes the proof of the second part of Proposition 3.112. □

3.7 Physical motivation: Symmetries of mechanical systems and Noether's principle

Let X be a manifold. Noether's principle states that every continuous symmetry of X that preserves the Lagrangian function gives rise to a conserved quantity (integral of motion). Via the Legendre transformation, this is a special case of the following remark about Hamiltonian mechanics: Let $H\colon M := T^*X \to \mathbb{R}$ be a smooth function (the Hamiltonian). A symmetry of the corresponding mechanical system is an action of a Lie group G on phase space M under which H is invariant. Assume that the action is Hamiltonian. We fix a momentum map $\mu\colon M \to \mathfrak{g}^*$.

Remark 3.114 (Noether's principle). For every $\xi \in \mathfrak{g}$, the function $\langle \mu, \xi \rangle \colon M \to \mathbb{R}$ is a constant (integral of motion) for H, i.e. $\langle \mu, \xi \rangle \circ \varphi_H^t = \langle \mu, \xi \rangle$ for all t.

Proof. By hypothesis, we have $H(\exp(t\xi)x) = H(x)$ for all $t \in \mathbb{R}$ and $x \in M$. Differentiating with respect to t we obtain $0 = \mathrm{d}H(X_\xi) = \mathrm{d}H(X_{\langle \mu, \xi \rangle}) = \{H, \langle \mu, \xi \rangle\}$. Therefore, the claimed equality follows from Proposition 3.52. □

Note: Integrals of motion are important in mechanics, as they can be used to decrease the degrees of freedom.

Example 3.115 (angular momentum)**.** Consider a particle in $\mathbb{R}^3$ subject to a conservative force. Its Hamiltonian is given by $H(q,p) = T(p) + U(q)$, where T is the kinetic energy and U is the potential energy. Assume that these functions are rotationally invariant, i.e. there exist functions $\widetilde{T}, \widetilde{U} \colon [0, \infty) \to \mathbb{R}$ such that $T = \widetilde{T}(|\_\_|)$ and $U = \widetilde{U}(|\_\_|)$. (For a Newtonian, i.e. non–relativistic, particle we have $T(p) = |p|^2/2m$). We claim that the angular momentum $q \times p$ is preserved (under the time evolution of the system). To see this, consider the standard action of SO(3) on $\mathbb{R}^3$ given by left–multiplication. The induced action of SO(3) on $T^*\mathbb{R}^3 = \mathbb{R}^3 \times (\mathbb{R}^3)^*$ is given by

$$\Phi(q,p) = (\Phi q, p\Phi^{-1}).$$

Hence it preserves the Hamiltonian H. By Example 3.109 the action is Hamiltonian, with the momentum map

$$\mu \colon \mathbb{R}^3 \times (\mathbb{R}^3)^* \to \mathfrak{so}(3)^*, \qquad \langle \mu(q,p), \xi \rangle = p\xi_{\mathbb{R}^3}(q) = p\xi q.$$

The Lie algebra $\mathfrak{so}(3)$ consists of all antisymmetric real 3×3 matrices. Hence the map $\Phi \colon \mathbb{R}^3 \to \mathfrak{so}(3)$

$$\Phi v := \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

is a vector space isomorphism. Identifying $\mathbb{R}^3$ with its dual via the standard inner product, we have $\mu(q,p)\Phi = q \times p$ (Assignment 12). Hence it follows from Remark 3.114 that the angular momentum $q \times p$ is a constant of motion for H, as claimed.

Remark 3.116. Under the canonical identification of $\mathbb{R}^3$ with its dual, the induced action of SO(3) on $T^*\mathbb{R}^3 = \mathbb{R}^6$ is given by the diagonal action $\Phi(q,p) = (\Phi q, \Phi p)$.

Remark 3.117. The term 'momentum map' is motivated by the above example and a similar example about linear momentum.

3.8 Symplectic Quotients

Symplectic quotients are defined for regular presymplectic manifolds, in particular for regular coisotropic submanifolds of symplectic manifolds. Let (M, ω) be a symplectic manifold and G a Lie group, and fix a Hamiltonian G–action on M and a momentum map $\mu \colon M \to \mathfrak{g}^*$. If 0 is a regular value then $\mu^{-1}(0)$ is a coisotropic submanifold of

M. Under suitable conditions the corresponding symplectic quotient is the set of orbits $\mu^{-1}(0)/G$ together with a canonical manifold structure and symplectic form induced by ω. To explain this, we need the following:

Definition 3.118. A *presymplectic manifold* is a pair (M, ω), where M is a manifold and ω is a closed 2–form on M of constant corank.

Example 3.119.

- Any manifold equipped with the 2–form 0.
- Any symplectic manifold is presymplectic.
- The product of two presymplectic manifolds.
- $(N, \iota^*\omega)$, where (M, ω) is a symplectic manifold and $\iota\colon N \to M$ is a coisotropic submanifold.

Definition 3.120. Let (M, ω) be a presymplectic manifold. We define the *isotropic leaf relation* to be the set

$$R^{(M,\omega)} = \{(x(0), x(1))\colon x \in C^\infty([0,1], M), \forall t\colon \dot{x}(t) \in T_{x(t)}M^\omega\}.$$

Remark 3.121. The set $R^{(M,\omega)}$ is an equivalence relation on M (Assignment 13). For a point $x \in M$ we define $M_x := M_x^\omega$, the isotropic leaf through x to be the $R^{(M,\omega)}$–equivalence class of x. We denote the set of these leafs by M_ω.

Remark 3.122. The subspaces $T_xM^\omega \subset T_xM$ form an involutive distribution on M, called the *isotropic* (or *characteristic*) distribution. By Frobenius' theorem such a distribution is integrable and hence gives rise to a foliation on M, called the isotropic foliation. The isotropic leafs are the leafs of this foliation.

Definition 3.123. We call (M, ω) regular if there exists a manifold structure on the set of isotropic leafs M_ω, such that the canonical projection $\pi\colon M \to M_\omega$ is a (smooth) submersion.

Theorem 3.124 (symplectic reduction)**.** *Let (M, ω) be a presymplectic manifold. Assume that it is regular. Then there exists a unique symplectic form ω_M on M_ω (wrt. the manifold structure from Definition 3.123) such that $\pi^*\omega_M = \omega$.*

Remark 3.125. Theorem 3.124 is a manifold version of the linear symplectic reduction of subsection 2.3. If it exists, then the manifold structure on M_ω as in this result is unique.

Definition 3.126. Assume that (M, ω) is regular. We define its symplectic quotient to be the triple $(M_\omega, \mathcal{A}, \omega_M)$, where $\mathcal{A}$ is the above manifold structure on M_ω.

An important class of examples of regular presymplectic manifolds is given as follows: Let (M, ω) be a symplectic manifold and G a Lie group. Fix a Hamiltonian action of G on M and a momentum map $\mu\colon M \to \mathfrak{g}^*$. Such a collection of data is called a *Hamiltonian manifold*. Since μ is G–equivariant, the action of G restricts to an action on $N = \mu^{-1}(0)$. We denote by $\iota\colon N \to M$ the inclusion.

Definition 3.127. An action of a topological group G on a topological space X is called *proper* if the map

$$G \times X \to X \times X, \qquad (g,x) \mapsto (x, gx)$$

is proper.

Example 3.128. Every continuous action of a compact group is proper.

Proposition 3.129. *Assume that G acts freely on $\mu^{-1}(0)$. Then the following holds:*

1. *0 is a regular value of μ and $N \coloneqq \mu^{-1}(0)$ is a coisotropic submanifold.*
2. *The isotropic leaf through a point $x \in N$ is the G–orbit through x.*
3. *If G also acts properly on $\mu^{-1}(0)$ then the pair $(N, \iota^*\omega)$ is a regular presymplectic manifold.*

Remark 3.130. By the last part of Example 3.119, under the hypotheses of this result, the pair $(N, \iota^*\omega)$ is a presymplectic manifold.

This result together with Theorem 3.124 has the following consequence. We denote by $\overline{M} = \mu^{-1}(0)/G$ the set of orbits in $\mu^{-1}(0)$ and by $\pi\colon \mu^{-1}(0) \to \overline{M}$ the canonical projection.

Corollary 3.131 (Marsden-Weinstein quotient)**.**

1. *If G acts freely and properly on $N \coloneqq \mu^{-1}(0)$ then there exists a manifold structure and a symplectic form on $\overline{M}$ such that π is a smooth submersion, and $\pi^*\overline{\omega} = \iota^*\omega$.*
2. *There exists at most one manifold structure on $\overline{M}$ for which π is a smooth submersion. For this structure there is at most one 2–form $\overline{\omega}$ on $\overline{M}$ satisfying $\pi^*\overline{\omega} = \iota^*\omega$.*

Under the hypotheses of this corollary we write $M/\!\!/G \coloneqq \mu^{-1}(0)/G$ for the symplecitc quotient. The idea behind this notation is that $M/\!\!/G$ is a 'doubly reduced space': First we reduce from M to $\mu^{-1}(0)$ and then to the quotient $\mu^{-1}(0)/G$. The dimension is thus reduced by twice the dimension of G. The following remark will be unsed in the next example:

Remark 3.132. Given a continuous and proper action of a topological group G on a topological space X and a G–invariant subset $Y \subset X$, the restricted action of G on Y is proper.

Example 3.133. Consider a smooth, free and proper action of a Lie group G on a manifold X. We define

$$\mu\colon T^*X \to \mathbb{R}, \qquad \langle \mu(q,p), \xi \rangle \coloneqq p(\xi_M)(q), \quad \forall \xi \in \mathfrak{g}$$

By Example 3.109 this is a momentum map for the induced action of G on T^*X. This action is free and proper. By Remark 3.132 the same holds for its restriction to $\mu^{-1}(0) \subset T^*X$. Hence $(\mu^{-1}(0), \iota^*\omega^{\text{can}})$ is a regular (by Proposition 3.129) presymplectic manifold. By Theorem 3.124 the symplectic quotient $\mu^{-1}/G = T^*X/\!\!/G$ is well–defined.

Example 3.134. Let $k, n \in \mathbb{N}$ and $G := U(k)$ act on $\mathbb{C}^{k\times n}$ by left–multiplication. We equip $\mathbb{C}^{k\times n} = \mathbb{R}^{2kn}$ with ω_0. WE define the inner product $\langle \_, \_\rangle$ on $\mathfrak{u}(k) = \mathrm{Lie}\, U(k)$ by $\langle \zeta, \zeta'\rangle := \mathrm{tr}(\zeta^*\zeta')$. By Remark 3.102 the map

$$\mu\colon \mathbb{C}^{k\times n} \to \mathfrak{u}(k)^*, \qquad \langle \mu(\Theta), \xi\rangle := \langle \tfrac{i}{2}(1 - \Theta\Theta^*), \xi\rangle$$

is a momentum map for the action. The zero–level set $\mu^{-1}(0)$ is the *Stiefel manifold*

$$V(k,n) := \{\Theta \in \mathbb{C}^{k\times n} \mid \Theta\Theta^* = 1\},$$

which can be interpreted as the set of orthonormal k–forms on $\mathbb{C}^n$.

Since $U(k)$ is compact, its action on $\mathbb{C}^{k\times n}$ is proper. Hence the same holds for its restriction to $V(k,n)$. This restricted action is also free. Hence $(V(k,n), \iota^*\omega_0)$ is a regular presymplectic manifold. Its sympletic quotient $V(k,n)/U(k) = \mathbb{C}^{k\times n}/\!\!/U(k)$ is the complex Grassmannian $G(k,n)$ of k–planes in $\mathbb{C}^n$. ($\Theta^*\mathbb{C}^k \subset \mathbb{C}^n$ is such a 'plane' for every $\Theta \in V(k,n)$ and $(U\Theta)^*\mathbb{C}^k = \Theta^*\mathbb{C}^k$. Also we obtain every k–plane in this way. Furthermore, if $\Theta\mathbb{C}^k = \Theta'\mathbb{C}^k$ and $\Theta, \Theta' \in V(k,n)$, then there exists $U \in U(k)$ such that $U\Theta = \Theta'$.) In the case $k = 1$ this is the complex projective space $\mathbb{C}P^{n-1}$ and the quotient symplecitc form is called the *Fubini Study form.* In the standard holomorphic chart

$$\mathbb{C}^{n-1} \ni z \to [1, z_1, \ldots, z_{n-1}] \in \mathbb{C}P^{n-1}$$

is given by $\frac{i}{2}\partial\overline{\partial}(\log(|z|^2 + 1))$, where

$$\partial := \sum_{i=1}^{n-1} \frac{\partial}{\partial z_i}\, \mathrm{d}z_i, \qquad \overline{\partial} = \sum_{i=1}^{n-1} \frac{\partial}{\partial \overline{z}_i}\, \mathrm{d}\overline{z_i}.$$

Remark 3.135. For $n = 2$ we get $\mathbb{C}P^1 = S^2 \subset \mathbb{R}^3$ with the standard symplectic form on S^2.

In the proof of Theorem 3.124 we will use

Remark 3.136. Let (M, ω) be a presymplectic manifold and $x_0 \in M$. We denote $n := \dim M$, $k := \mathrm{co\,rk}\,\omega = \mathrm{rk}\, TM^\omega$ and $2m = \mathrm{rk}\,\omega$. By an exercise in Assignment 13 the isotropic distribution TM^ω is involutive and hence integrable, i.e. there exists a foliation chart around x_0, or equivalently a local parametrization $\psi\colon \mathbb{R}^n \to M$ satisfying $\psi(0) = x_0$ and $\mathrm{d}\psi(x')(\{0\} \times \mathbb{R}^k) = (T_{\psi(x')}M)^\omega$ for all $x' \in \mathbb{R}^n$. This implies that $\pi \circ \psi(x_1', x_2') = \pi \circ \psi(x_1', y_2')$ for all $x_1' \in \mathbb{R}^{2m}$ and $x_2', y_2' \in \mathbb{R}^k$. It follows that $\mathrm{d}(\pi \circ \psi)(x_1', x_2')v' = \mathrm{d}(\pi \circ \psi)(x_1', y_2')v' \in T_{\pi\circ\psi(x_1', x_2')}M_\omega$ for all $x_1' \in \mathbb{R}^{2m}$, $x_2', y_2' \in \mathbb{R}^k$ and $v' \in \mathbb{R}^n$.

Proof of Theorem 3.124. We prove existence of ω_M. Let $\overline{x} \in M_\omega$ and $\overline{v}_1, \overline{v}_2 \in T_{\overline{x}}\overline{M}$. We choose a point $x \in \overline{x}$ and vectors $v_1, v_2 \in T_xM$ satisfying $\mathrm{d}\pi(x)v_i = \overline{v}_i$. We define $(\omega_M)_x(\overline{v}_1, \overline{v}_2) := \omega_x(v_1, v_2)$. By Assignment 13 we have $\ker \mathrm{d}\pi(x) = (T_xM)^{\omega_x}$, and therefore $\omega_x(v_1, v_2)$ does not depend on the choices of v_i. We show that it does not depend on the choice of $x \in \overline{x}$: Consider first the case in which $M = \mathbb{R}^n$ together with the standard foliation, which corresponds to the distribution whose fiber at $x \in \mathbb{R}^n =$

$\mathbb{R}^{2m} \times \mathbb{R}^k$ is given by $\{0\} \times \mathbb{R}^k$. Let $x_0, x_1 \in \overline{x}$. By definition there exists a smooth path $x\colon [0,1] \to \mathbb{R}^n$ satisfying $x(i) = x_i$ for $i \in \{0,1\}$ and $\dot{x}(t) \int (T_{x(t)}\mathbb{R}^n)^\omega = \{0\} \times \mathbb{R}^k$. Denoting by $\mathrm{pr}\colon \mathbb{R}^n = \mathbb{R}^{2m} \times \mathbb{R}^k \to \mathbb{R}^{2m}$ the canonical projection, it follows that $\mathrm{pr}(x_1) = \mathrm{pr}(x_0)$. We write

$$\omega = \sum_{i<j} \omega_{ij} \,\mathrm{d}x^i \wedge \mathrm{d}x^j.$$

Our assumption that $(T_x\mathbb{R}^n)^\omega = \{0\} \times \mathbb{R}^k$ implies that $\omega_{ij} = 0$, if i or j is in $\{2m + 1, \ldots, n\}$. By hypothesis, we have

$$0 = \mathrm{d}\omega = \sum_{i<j,l} \partial_l \omega_{ij} \,\mathrm{d}x^l \wedge \mathrm{d}x^i \wedge \mathrm{d}x^j.$$

It follows that $\partial_l \omega_{ij} = 0$ for all $l \in \{2m+1, \ldots, n\}$ and $i < j$. It follows that $\omega_{x_0} = \omega_{x_1}$. For $i \in \{1,2\}$ let $\overline{v}_i \in T_{\overline{x}}(\mathbb{R}^n_\omega) = \mathbb{R}^n/(\{0\} \times \mathbb{R}^k)$. We choose vectors $v_i \in \mathbb{R}^n$ such that $\mathrm{d}\pi(x_0)v_i = v_i + (\{0\} \times \mathbb{R}^k) = \overline{v}_i$. By the definition of the standard foliation, it follows that $\mathrm{d}\pi(x_1)v_i i = v_i + (\{0\} \times \mathbb{R}^k) = \overline{v}_i$. We also have $\omega_{x_0}(v_1, v_2) = \omega_{x_1}(v_1, v_2)$. Combining this, it follows that the right hand side of the equation $(\omega_M)_x(\overline{v}_1, \overline{v}_2) = \omega_x(v_1, v_2)$ does not depend on the choice of representative of $\overline{x}$ in the case $M = \mathbb{R}^n$ with the standard foliation.

In the general situation let again $\overline{v}_1 \in T_{\overline{x}}(M_\omega)$ and $x_0, x_1 \in \overline{x}$. Consider the case in which there exists a smooth path $x\colon [0,1] \to M$ tangent to TM^ω and connecting x_0 and x_1, and a foliation paramatrization ψ as in Remark 3.136 satisfying $x([0,1] \subset U = \psi(\mathbb{R}^n)$. Recall that this means that $\psi(0) = x_0$ and $\mathrm{d}\psi(x')(\{0\} \times \mathbb{R}^k) = (T_{\psi(x')}M)^\omega$ for all $x' \in \mathbb{R}^n$.

The path $x' \coloneqq \psi^{-1} \circ x\colon [0,1] \to \mathbb{R}^n$ is tangent to the standard distribution with fiber $\{0\} \times \mathbb{R}^k$ and therefore satisfies $\mathrm{pr} \circ x'(0) = \mathrm{pr} \circ x'(1)$. We choose vectors $v_i^0 \in T_{x_0}M$ satisfying $\mathrm{d}\pi(x_0^i) = \overline{v_i}$. We define

$$v_i \coloneqq \mathrm{d}(\psi^{-1})(x_0)v_i^0, \qquad v_i^1 \coloneqq \mathrm{d}\psi(x'(1))v_i'$$

By Remark 3.136, we have

$$\mathrm{d}\pi(x_1)v_i^1 = \mathrm{d}(\pi \circ \psi)(x'(1))v_i' = \mathrm{d}(\pi \circ \psi)(x'(0))v_i' = \mathrm{d}\pi(x_0)v_i^0 = \overline{v}_i.$$

Furthermore, the isotropic foliation for $\psi^*\omega$ is the standard foliation. (Check this!) Therefore, by what we already proved, we have

$$\omega_{x_0}(v_1^0, v_2^0) = (\psi^*\omega)_{x'(1)}(v_1', v_2') = (\psi^*\omega)_{x'(1)}(v_1', v_2') = \omega_{x_1}(v_1^1, v_2^1).$$

This shows that $(\omega_M)_{\overline{x}}$ is well–defined in the case in which there exist $x\colon [0,1] \to M$ and ψ as above. In the general situation we choose a finite collection of foliation charts that cover the image of $x([0,1]) \subset M$. The statement then follows from an induction argument.

We show that ω_M is clsed and non–degenerate: Let $\overline{x} \in M_\omega$. Since $\pi\colon M \to M_\omega$ is a submersion, by an argument using the Implicit Function Theorem there exists an open neighbourhood $U \subset M_\omega$ of $\overline{x}$ and a smooth map $f\colon U \to M$ satisfying $\pi \circ f = \mathrm{id}$. (Compare to Assignment 8 and Assignment 13). Such a map f is called a *(local) slice.*It follows that $\omega_M|_U = f^*\pi^*\omega_M = f^*\omega$ and therefore $(\mathrm{d}\omega_M)|_U = f^*\,\mathrm{d}\omega = 0$. This proves that ω_M is closed.

We show that it is non–degenerate. We denote $x := f(\overline{x})$. By Assignment 13 we have $\ker \mathrm{d}\pi(x) = (T_xM)^\omega$ and therefore $\mathrm{im}\,\mathrm{d}f(\overline{x}) \cap (T_xM)^\omega = \mathrm{im}\,\mathrm{d}f(\overline{x}) \cap \ker \mathrm{d}\pi(x) = 0$. The last equality follows from the fact that $\mathrm{d}\pi(x)\,\mathrm{d}f(\overline{x}) = \mathrm{d}(\pi \circ f)(\overline{x}) = \mathrm{id}$. Since $\dim M_\omega + \mathrm{co\,rk}\,\omega = \dim M$, it follows that $\mathrm{im}\,\mathrm{d}f(\overline{x}) + (T_xM)^\omega = T_xM$. It follows that $\mathrm{d}f(\overline{x})(T_{\overline{x}}M_\omega)^{\omega_M} \subset (T_xM)^\omega = \ker \mathrm{d}\pi(x)$ and therefore, using injectivity of $\mathrm{d}f(\overline{x})$, we have $(T_{\overline{x}}M_\omega)^{\omega_M} = 0$. Hence ω_M is non–degenerate.

Uniqueness of the symplecic form on M_ω follos from Assignment 13. □

Remark 3.137. Let (M, ω, G, μ) be a Hamiltonian manifold. Assume that the action of G on $N := \mu^{-1}(0)$ is free and proper. We denote by $\iota\colon N \to M$ the inclusion. By Proposition 3.129 the pair $(N, \iota^*\omega)$ is a regular presymplecitc manifold. In this situation, there is an alternative argument showing that the symplectic form on the quotient $\overline{M} := \mu^{-1}(0)/G$ does not depend on the choice of a representative $x \in \mu^{-1}(0)$ of the given point $\overline{x} \in \overline{M}$. Namely, this follows from G–invariance of ω.

The proof of the third part of Proposition 3.129 is based on

Theorem 3.138 (Slice theorem)**.** *Let M be an n–dimensional manifold, $x \in M$ and let a k–dimensional Lie group G act smoothly, freely and properly on M. There exists a G–invariant embedding $\psi\colon \mathbb{R}^{n-k} \times G \to M$ satisfying $\psi(0, g) = gx$ for all $g \in G$.*

Proof. This follows from V. Guillemin, V. Ginzburg, Y. Karshon 'Moment maps, cobordisms, and Hamiltonian group actions', 2002, Theorem B.24 on page 180, or T. tom Dieck, 'Transformation grups', 1987, Theorem (5.6) on page 40 (for compact G). □

Remark 3.139. The reason for the name 'Slice Theorem' is that the image $\psi(\mathbb{R}^{n-k} \times \{0\})$ is a slice for the action of G on M, i.e. a submanifold of M of codimension k that is transverse to the orbits of G and intersects every orbit at most in one element.

Corollary 3.140. *Under the hypotheses of Theorem 3.138 there exists a manifold structure on the set of orbits M/G such that the canonical projection $\pi\colon M \to M/G$ is a submersion.*

Proof. Let $\iota\colon \mathbb{R}^{n-k} \to \mathbb{R}^{n-k} \times G$ be defined by $\iota(y) = (y, 1)$. We show that the set

$$\overline{\mathcal{A}} := \{\pi \circ \psi \circ \iota \mid \psi\colon \mathbb{R}^{n-k} \times G \to M \text{ is an equivariant embedding}\}$$

is an atlas on M/G. It follows from Theorem 3.138 that $\overline{\mathcal{A}}$ covers M/G, i.e. $\bigcup \overline{\psi}(\mathbb{R}^{n-k}) = M/G$, where $\overline{\psi} := \pi \circ \psi \circ \iota$. Let $\overline{\psi}, \overline{\psi}' \in \overline{\mathcal{A}}$. We choose equivariant embeddings $\psi, \psi'\colon \mathbb{R}^{n-k} \times G \to M$ such that $\overline{\psi} = \pi \circ\ \psi \circ \iota$. We denote by $\mathrm{pr}\colon \mathbb{R}^{n-k} \times G \to \mathbb{R}^{n-k}$ the canonical projection. The transition map from $\overline{\psi}$ to $\overline{\psi}'$ is

$$\overline{\psi}'^{-1} \circ \overline{\psi} = \mathrm{pr} \circ \psi'^{-1} \circ \psi \circ \iota\colon \overline{\psi}^{-1}(\mathrm{im}(\overline{\psi}')) \subset \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}.$$

This map is a diffeomorphism onto its image. Hence the altas $\overline{\mathcal{A}}$ is compatible. By an elementary argument the induced topology is Hausdorff and second countable. □

Proof of Proposition 3.129. For the first and second statement see Assignment 13. The third statement is an immediate consequence of Corollary 3.140. □