Vorlesung aus dem Wintersemester 2012/13

Differential Geometry

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1 Manifolds and differential forms

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1.1 Reminder from calculus in several variables

Differentiability. The map $F : U \longrightarrow \mathbb{R}^n$ is called differentiable in $x_0 \in U$ if F is near x_0 well approximable by a linear map in the following sense: There exists a linear map $L : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ s.t. $F(x_0 + h) = F(x_0) + L(h) + r(h)$ with an error r (defined in a neighbourhood) which is of smaller order than h,

$$\lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = 0.$$

We also write $F(x_0 + h) = F(x_0) + L(h) + o(\|h\|)$.

If F is differentiable in x_0 then L is uniquely determined and called the differential of F in x_0

$$dF(x_0) \in \text{Hom}(\mathbb{R}^m, \mathbb{R}^n).$$

If for $v \in \mathbb{R}^m$ the limit $\partial_v F(x_0) := \lim\limits_{t \to 0} \frac{F(x_0+tv)-F(x_0)}{t}$ exists, it is called directoral derivative of F in direction of v. Partial derivatives are defined as

$$\frac{\partial F}{\partial x_i}(x_0) := \partial_{e_i} F(x_0).$$

if F is differentiable in x_0, then all directorial derivatives in x_0 exist and $\partial_v F(x_0) = dF_{x_0}(v)$. Relative to the standard basis (of $\mathbb{R}^m$ and $\mathbb{R}^n$) the differential $dF(x_0)$ is given by the Jacobian matrix

$$\left(\frac{\partial F_i}{\partial x_j}\right)_{i \leq n; j \leq m}$$

If $n = 1$, i.e. if F is a real valued function, then $dF_{x_0} \in \text{Hom}(\mathbb{R}^m, \mathbb{R}) = (\mathbb{R}^m)^*$ is al linear form and corresponds via the standart scalar product $\langle \bullet, \bullet \rangle$ on $\mathbb{R}^m$ to the gradient $\nabla F(x_0)$ of F in x_0:

$$dF_{x_0} = \langle \nabla F(x_0), \bullet \rangle$$

Warning: partial differentiability $\not\Rightarrow$ total differentiability. However, if the partial derivatives exist near x_0 and are continuous in x_0, then the map is differenatiable in x_0. In particular, the following statements are equivalent:

a) F is partially differentiable on U and the partial derivatives are continuous

b) F is differentiable on U and the differentail is continuous as map $U \longrightarrow \text{Hom}(\mathbb{R}^m, \mathbb{R}^n) \cong \mathbb{R}^{nm} : x_0 \longrightarrow dF(x_0)$.

Such maps are called continuously differentiable or of class C^1. More generally, one calls k times continuously differentiable maps of class $C^k, k \in \mathbb{N}_0 \cup \{\infty\}$ (C^0 is just continuous).

A C^k diffeomorphism between open sets $V, W \subset \mathbb{R}^m$ is a bijective C^k-map $F : V \longrightarrow W$ whose inverse $F^{-1} : W \longrightarrow V$ is also a C^k-map (if $k = 0$: homeomorphism).

A map $F : U \longrightarrow \mathbb{R}^m, U \subset \mathbb{R}^m$ open, is called a local C^k-diffeomorphism in x_0, if there are open neighbourhoods V of x_0 in U and W of $F(x_0)$ in $\mathbb{R}^m$ s.t. $F_{\restriction V} : V \longrightarrow W$ is a C^k-diffeomorphism.

We observe: F is a local diffeomorphism in $x_0 \Rightarrow dF_{x_0}$ is invertible (as a linear map), because by the chain rule:

$$dG_{y_0} \circ dF_{x_0} = d(\underbrace{G \circ F}_{\mathrm{id}_{\mathbb{R}_m}})_{x_0} = \mathrm{id}_{\mathbb{R}^m}$$

Inverse Function Theorem (Umkehrsatz): Suppose that $F : U \longrightarrow \mathbb{R}^m$ is $C^{k \geq 1}, U \subset \mathbb{R}^m$ open, $x_0 \in U$. If dF_{x_0} is invertible, then F is a local C^k - diffeo.

Remark. dF_{x_0} invertible $\Leftrightarrow$ $\det \frac{\partial F_i}{\partial x_j}(x_0) \neq 0$.

Clever application of the Inverse Function Theorem yields as a corollary the following generalization:

Implicit Function Theorem: Suppose that $U \subseteq \mathbb{R}^m$ is open, m $\geq$ n and $F : U \longrightarrow \mathbb{R}^n$ is $C^{k \geq 1}$. Furthermore let $\mathbb{R}^m = E_1^{m-n} \bigoplus E_2^n$ a direct sum decomposition into linear subspaces with $\dim E_1 = m - n$, $(x_0 \in E_1, y_0 \in E_2) \in U$ and $Z_0 := F(x_0, y_0)$. If $dF(x_0, y_0)_{\restriction E_2} : E_2 \longrightarrow \mathbb{R}^n$ is invertible, then there exist open neighbourhoods V_1 of x_0 in E_1, V_2 of y_0 in E_2 and W of z_0 in $\mathbb{R}^n$ and for each $z \in W$ a C^k-map $G_z : V_1 \longrightarrow V_2$ s.t. $\underbrace{F^{-1}(z) \cap V_1 \times V_2}_{\text{solutions of } F(x,y)=z \text{ near } (x_0,y_0)} = \{\underbrace{(x, G_z(x))}_{\text{graph of } G_z} / x \in V_1\}$. Furthermore, G_z depends C^k-differentiably on z, i.e. the map $V_1 \times W \longrightarrow \mathbb{R}^m : (x, z) \longrightarrow G_z(x)$ is C^k- differentiable, more precisely a C^k-diffeo onto an open neighbourhood of (x_0, y_0), with $G_{z_0}(x_0) = y_0$.

Remark.

i) Near (x_0, y_0) the function/map G_z is implicitly given by the equation $F(x, G_z(x)) = 0$

ii) Let $p \in U$. There exists a decomposition $\mathbb{R}^m = E_1 \bigoplus E_2$ s.t. $dF_{p \restriction E_2}$ is invertible and dF_p is surjective.

Remark.

Let $p_0 \in U$. A decomposition $\mathbb{R}^m = E_1 \oplus E_2$ such that $\mathrm{d}F_{p_0}|_{E_2}$ is invertible, exists iff $\mathrm{d}F_{p_0}$ is surjective.

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Definition 1.1. Let $F \colon \mathbb{R}^m \supset U \to \mathbb{R}^n$ with U open be $\mathscr{C}^k$-differentiable.

1. A point $x \in U$ is called *regular* if $\mathrm{d}F_x$ is surjective, and *critical* or *singular* otherwise.

2. A value $y \in \mathbb{R}^n$ is called *regular* if all points $x \in F^{-1}(y)$ are regular, and *critical* or *singular* otherwise.

Remark.

1. Values which are not attained are trivially regular.
2. Sard's theorem says that the set of critical values has Lebesgue measure zero in $\mathbb{R}^n$.

Definition 1.2.

1. F is called a *submersion*, if $\mathrm{d}F_x$ is surjective for all $x \in U$, i.e. all points are regular, or equivalently, all values are regular.
2. F is called an *immersion* if $\mathrm{d}F_x$ is injective for all $x \in U$.

As a consequence of the implicit function theorem, for $y \in \mathbb{R}^n$, the set of solutions of the equation $F(x) = y$ is in every regular point locally a graph. In particular, it is locally parametrizable by independent coordinates. This leads us to the notion of a submanifold of Euclidean space.

1.2 Submanifolds of Euclidean space

Submanifolds of $\mathbb{R}^n$ are subsets which are regular in the sense that they can be made flat locally by a suitable coordinate change, i.e. be transformed into an affine subspace. In particular, they can be locally parametrized by independent coordinates.

Definition 1.3. Let $0 \leq d \leq n$, $k \geq 1$. A subset $M \subset \mathbb{R}^n$ is called a d-dimensional *differentiable submanifold* of class $\mathscr{C}^k$, if for every point $x \in M$ there exists a $\mathscr{C}^k$-diffeomorphism $\phi \colon U \to V$ from an open neighborhood U of x in $\mathbb{R}^n$ onto an open subset $V \subset \mathbb{R}^n$, such that

$$\phi(M \cap U) = \left(\mathbb{R}^d \times \{0\}\right) \cap V$$

Example.

1. The unit sphere
$$S^{n-1} = \left\{ x \in \mathbb{R}^n \middle| \sum_{i=1}^n x_i^2 = 1 \right\}$$
is covered by the open (relative to the subspace topology) subsets $S^n \cap \{x | x_k \gtrless 0\}$. These can be made flat, for instance by the map
$$\phi(x) = x \mp \left(1 - \sum_{i \neq k} x_i^2\right)^{1/2} e_k$$
2. The following subsets are not submanifolds in $\mathbb{R}^2$:

We now give two characterizations of submanifolds. First, we characterize them locally as solution sets.

Theorem 1.4. *The inverse images of regular values of $C^{k\geq 1}$-maps $F\colon \mathbb{R}^n \supset U \to \mathbb{R}^m$ are $\mathscr{C}^k$-submanifolds of dimension $n-m$, i.e. codimension m.*

Proof. Consequence of the implicit function theorem.

Example.

1. The unit sphere S^{n-1} is a $\mathscr{C}^\infty$-submanifold since it is the inverse image of 1 under the $\mathscr{C}^\infty$-map $F(x) = \|x\|^2$.

2. Consider
$$\det\colon \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^\times$$
where $\mathrm{GL}(n, \mathbb{R})$ is viewed as an open subset of $\mathbb{R}^{n\times n}$. det is smooth, since it is polynomial. We have (exercise)
$$(\,\mathrm{d}\det)_E = \mathrm{tr}\,.$$
Hence $\mathrm{d}\det_E$ is surjective and E is a regular point for the determinant. With the multiplicativity of det follows that all $A \in \mathrm{GL}(n, \mathbb{R})$ are regular points for det, because differentiating $\det(AX) = \det(A)\det(X)$ with respect to X in E in direction V yields
$$(\,\mathrm{d}\det)_A(AV) = \det(A)(\,\mathrm{d}\det)_E(V) = \det(A)\,\mathrm{tr}(V)$$
so $\mathrm{d}\det_A = \det A \cdot \mathrm{tr}(A^{-1}-)$, which is surjective. So all points and values are regular, i.e. $\det\colon \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}^\times$ is a surjective submersion. In particular, 1 is a regular value which implies that $\mathrm{SL}(n, \mathbb{R}) = \det^{-1}(1)$ is a smooth submanifold of codimension 1.

3. To show that the orthogonal group
$$\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) | AA^t = E\}$$
is a submanifold, consider
$$F\colon\ \mathrm{GL}(n, \mathbb{R}) \to \mathrm{Mat}_{\mathrm{sym}}(n \times n, \mathbb{R}) \cong \mathbb{R}^{n(n+1)/2}, X \mapsto XX^t.$$
Then $\mathrm{d}F_X(H) = HX^t + XH^t$ and $\mathrm{d}F_X(E) = H + H^t$. So E is a regular point for F, $F(E) = E$, and thus $\mathrm{O}(n) = F^{-1}(E)$ is a $\mathscr{C}^\infty$-submanifold locally at E, i.e. there exists an open neighborhood U of E in $\mathrm{GL}(n, \mathbb{R})$ such that $\mathrm{O}(n) \cap U$ is a $\mathscr{C}^\infty$-submanifold. Due to the homogeneity, $\mathrm{O}(n)$ is in every point A locally a submanifold, because left multiplication $L_A\colon X \to AX$ is a $\mathscr{C}^\infty$-diffeomorphism of $\mathrm{GL}(n, \mathbb{R})$ and preserves $\mathrm{O}(n)$. So $\mathrm{O}(n)$ is a $\mathscr{C}^\infty$)-submanifold with dimension $n^2 - n(n+1)/2$.

Second, we characterize manifolds locally by parametrizability.

Definition 1.5. A d-dimensional local $\mathscr{C}^k$-parametrization of M near p is a map

$$V' \xrightarrow{F} U' = M \cap U \ni p$$

where V', U' are open in M and F is a homeomorphism and a $\mathscr{C}^k$-immersion.

Remark. In general, injective immersions are no homeomorphisms onto their image (topological embeddings).

Example. Consider the curve

$$\begin{aligned} (-1,\infty) &\xrightarrow{\quad c \quad} \mathbb{R}^2 \\ t &\longmapsto \left(\tfrac{t}{1+t^3}, \tfrac{t^2}{1+t^3}\right). \end{aligned}$$

It is injective and C^∞-smooth, but its inverse is not continuous.

For a subset $M \subset \mathbb{R}^n$ holds: M is a submanifold if and only if there exists a local parametrization. (One direction is trivial, the other is given by the Inverse Function Theorem).

Proposition 1.6. *Let $M \subset \mathbb{R}^n$. M is a submanifold of $\mathbb{R}^n$ if and only if there exists a local parametrization.*

Proof. We start with a local parametrization F near p, $F(0) = p$. We thicken F to a $\mathscr{C}^k$-differentiable map

$$V' \times (-\varepsilon, \varepsilon)^{n-d} \xrightarrow{\ \overline{F}\ } \mathbb{R}^n$$

such that $\mathrm{d}\overline{F}_0$ is surjective. Then the Inverse Function Theorem yields that $\overline{F}$ is locally invertible at 0, i.e. there is a neighborhood V_1 of 0 such that $\overline{F}|_{V_1}$ is a $\mathscr{C}^k$-diffeomorphism onto $\overline{F}(V_1)$. We have

$$\overline{F}(\mathbb{R}^d \times \{0\} \cap V_1) \subset M \cap U_1$$

To achive equality, we shrink V_1 (and U_1). Since F is a homeomorphism onto its image, $F(\mathbb{R}^d \times \{0\} \cap V_1)$ is open in M, so it can be written in the form $M \cap U_2$, where U_2 is an open subset of U_1. Define $V_2 := (\overline{F}|_{V_1})^{-1}(U_2)$. Then

$$\overline{F}(\mathbb{R}^d \times \{0\} \cap V_2) = F(\mathbb{R}^d \times \{0\} \cap V_1) \cap \overline{F}(V_2) = (M \cap U_1) \cap U_2 = M \cap U_2.$$

So $(\overline{F}|_{V_2})^{-1}$ makes M flat near p, i.e. M is a submanifold. ☐

Example. Consider the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$.

1. The local parametrization results from S^{n-1} being locally a graph. For example,

   $$\varphi\colon \{y \in \mathbb{R}^{n-1} | \|y\| < 1\} \to S^{n-1}, y \mapsto (y, (1 - \|y\|^2)^{1/2})$$

   is a parametrization of the upper hemisphere: Its image is the open subset $S^{n-1} \cap \{x_n > 0\}$ of S^{n-1} and it is inverted by the projection onto the first $n-1$ components.

2. Another way to parametrize the sphere is the *stereographic projection*: It is given by the formula

$$\sigma_N\colon S^{n-1}\smallsetminus\{N\}\to\mathbb{R}^{n-1}, x\mapsto\frac{1}{1-x_n}(x_1,\dots,x_{n-1})$$

where $N=(0,\dots,0,1)$ is the north pole of the spere. Its inverse is given by

$$y\mapsto\frac{1}{\|y\|^2+1}(2y_1,\dots,2y_{n-1},\|y\|^2-1).$$

Geometrically, it projects a point $x\in S^{n-1}\smallsetminus\{N\}$ to the unique point $y\in\mathbb{R}^{n-1}$ such that x, N, and $(y,0)$ lie on one line.

Analogously, one can define the projection

$$\sigma_S\colon S^{n-1}\smallsetminus\{S\}\to\mathbb{R}^{n-1}$$

where $S=(0,\dots,0,-1)$ is the south pole of the sphere.

We summerize:

Theorem 1.7. *For $M\subset\mathbb{R}^n$ are equivalent:*

1. *M is a d-dimensional submanifold.*
2. *M is locally the image of a regular value of a $\mathbb{R}^{n-d}$-valued $\mathscr{C}^k$-map.*
3. *There exist local d-dimensional $\mathscr{C}^k$-parametrizations of M.*

The system of local parametrizations forms the *differentiable structure* on M, which enables us to do Analysis (differentiate etc.) intrinsically (without reference to the ambient space).

Let M be a d-dimensional submanifold of $\mathbb{R}^n$, $p\in M$, and $\phi\colon U\to V$ the map that locally flattens M. Then the restriction of ϕ to $M\cap U$ is a homeomorphism onto an open subset V' of $\mathbb{R}^d$. It is called a *local chart* or *local coordinate map*. It is the inverse of a local parametrization. We observe that the coordinate changes are $\mathscr{C}^k$: Let $U_1\cap M$ and $U_2\cap M$ be two open subsets of M (U_1,U_2 are open in $\mathbb{R}^n$). The two $\mathscr{C}^k$-diffeomorphisms ϕ_1,ϕ_2 which flatten M map U_1,U_2 to open subsets $\phi_1(U_1),\phi_2(U_2)\subset\mathbb{R}^n$. If $U_1\cap U_2\neq\varnothing$, the coordinate changes are given by restrictions of $\phi_2\circ\phi_1^{-1}$ and $\phi_1\circ\phi_2^{-1}$:

$$\phi_2\circ\phi_1^{-1}\colon\mathbb{R}^d\times\{0\}\cap\phi_1(U_1\cap U_2)\to\mathbb{R}^d\times\{0\}\cap\phi_2(U_1\cap U_2)$$

$$\phi_1\circ\phi_2^{-1}\colon\mathbb{R}^d\times\{0\}\cap\phi_2(U_1\cap U_2)\to\mathbb{R}^d\times\{0\}\cap\phi_1(U_1\cap U_2)$$

These are mutually inverse $\mathscr{C}^k$-diffeomorphisms.

This *compatibility* of local charts makes it possible e.g. to define differentiability of functions $f\colon M\to\mathbb{R}$ which are given only on M.

Definition 1.8. A function $f\colon M \to \mathbb{R}$ is called *l-times differentiable* in $x \in M$, $l \leq k$, if for some chart κ around x the function $f \circ \kappa^{-1}$ defined on the open subset $\operatorname{Im}(\kappa)$ is l-times differentiable in $\kappa(x)$. This is independent of the chosen chart κ. Indeed, if κ' is another chart around x, then

$$f \circ \kappa'^{-1} = (f \circ \kappa^{-1}) \circ (\kappa \circ \kappa'^{-1})$$

Since the coordinate change $\kappa \circ \kappa'^{-1}$ is a $\mathscr{C}^k$-diffeomorphism, $f \circ \kappa'^{-1}$ is $\mathscr{C}^k$-differentiable if and only if $f \circ \kappa^{-1}$ is. This definition of differentiability is intrinsic since it does not refer to the ambient Euclidean space.

This leads us to the notion of abstract manifolds, i.e. manifolds which are not a priori embedded into Euclidean space).

1.3 Abstract differentiable manifolds

Definition 1.9. A topological space $(X, \mathcal{T})$ is called *d-dimensional locally Euclidean* if every point has a local neighborhood which is homeomorphic to an open subset of $\mathbb{R}^d$.

In particular, there exist local parametrizations by independent coordinates.

Example.

1. $\mathscr{C}^0$-submanifolds of $\mathbb{R}^n$ as abstract topological spaces are locally Euclidean.
2. The eight and the X (see above) are not locally Euclidean.

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Example of an "abstract" locally euclidean space which does not arise in a natural way by a distinguished embedding into an euclidean space: real projective space:

$$\mathbb{RP}^d := \{1 - \text{dim linear subspaces of } \mathbb{R}^{d+1}\}$$

To a point $\mathbb{R}_x$ represented by $x = (x_0, x_1, \dots x_n) \in \mathbb{R}^{d+1} - \{0\}$ one assigns the homogeneous coordinates $[x] = [x_0 : \cdots : x_d]$. They are unique up to scaling

$$[x] = [x'] \Leftrightarrow \exists \lambda \in \mathbb{R}^* \text{with } x' = \lambda x.$$

One equippes $\mathbb{RP}^d$ with the quotient topology with respect to the projection π :

$$\begin{aligned} \mathbb{R}^{d+1} - \{0\} &\longrightarrow \mathbb{RP}^d \\ x &\longrightarrow \mathbb{R}x = [x]. \end{aligned}$$

Open sets in $\mathbb{RP}^d$ correspond to open double cones in $\mathbb{R}^{d+1}$. Note that π is an open map. Restricted to the unit sphere $S^d \subset \mathbb{R}^{d+1} - \{0\}$, the projection corresponds to dividing out the antipodal involution $x \longrightarrow -x$.

$$S^d \xleftrightarrow{inclusion} \mathbb{R}^{d+1} - \{0\} \xrightarrow{\pi} \mathbb{RP}^d$$

$$S^d \xrightarrow[2:1]{\pi/S^d} \mathbb{RP}^d$$

$$S^d \xrightarrow[2:1]{x \to \{x, -x\}} S^d / \pm 1 \xrightarrow[1:1]{\{x,-x\} \to \mathbb{R}x = [x]} \mathbb{RP}^d$$

Also $\pi|_{S^d}$ is open, because for small $r > 0$ and $z \in \mathbb{R}^{d+1}$ with $\|z\| = \sqrt{(1+x^2)}$ holds

$$\pi|_{S^d}(B(z,r) \cap S^d) = \pi(\underbrace{B(z,r)}_{\text{open in } \mathbb{R}^{d+1}-\{0\}}) \quad \text{is open in } \mathbb{RP}^d\text{, because } \pi \text{ open.}$$

$\pi|_{S^d}$ is continuous, open, locally injective (e.g. injective on open hemispheres) $\Rightarrow \pi|_{S^d}$ is local homeo $\overset{S^d \text{loc.eucl.}}{\Rightarrow} \mathbb{RP}^d$ is local euclidean.
A homeomorphism $U \xrightarrow{\kappa} \kappa(U)$ of an open subset $U \subseteq X$ onto an open subset $\kappa(U) \subseteq \mathbb{R}^d$ is called a local chart or set of local coordinates. U is called the domain of the chart.
If $(U_i, \kappa_i), i = 1,2$ are two charts, then the coordinate change $\kappa_1(U_1 \cap U_2) \xrightarrow{\kappa_2 \circ \kappa_1^{-1}} \kappa_2(U_1 \cap U_2)$ is a homeo between (possibly empty) open subsets of $\mathbb{R}^d$.
An atlas is a family (A) of charts $(U_i, \kappa_i), i \in I$ (index set), which cover X, $X = \bigcup_{i \in I} U_i$.
Two charts are C^k- compatible if the coordinate changes in boths directions are C^k-differentiable and hence C^k-diffeomorphisms. An atlas is called C^k-differentiable if any two of its charts are C^k-compatible. A C^k-differentiable atlas is contained in a unique maximal C^k-differentiable atlas which arises by adding all C^k-compatible charts.

Definition 1.10. A C^k- differentiable structure, $1 \leq k \leq \infty$, on a loc. eucl. space is a max C^k- differentiable atlas.

Example:

1) If $M \subset \mathbb{R}^n$ is a C^k-submanifold, then the local parametrizations, $1 \leq l \leq k$, form a C^k-differentiable structure on M, the natural one induced by $\mathbb{R}^n$.

2) $S^{n-1} = \{\|x\| = 1\} \subset \mathbb{R}^n$: The natural C^∞-differentiable structure induced by $\mathbb{R}^n$ is generated by an atlas consisting only of two charts, namely the sterogrphic projections

$$\kappa_\pm : S^{n-1} - \{\pm e_n\} \longrightarrow \mathbb{R}^{n-1}, \quad x \longrightarrow \frac{1}{1 \mp x_n}(x_1, \dots x_{n-1}).$$

We compute the coordinate change:

$$y = \frac{1}{1+x_n}(x_1, \dots x_{n-1}) \xrightarrow{\kappa_-^{-1}} x \xrightarrow{\kappa_+} \frac{1}{1-x_n}(x_1, \dots x_{n-1}) = \frac{1}{\|y\|^2} y$$
$$\|y\|^2 = \frac{1-x_n^2}{(1+x_n)^2} = \frac{1-x_n}{1+x_n}$$

3) $\mathbb{RP}^d$. Consider for $i = 0, \dots, d$ the bijection

$$U_i := \{[x] \in \mathbb{RP}^d / x_i \neq 0\} \xrightarrow{\kappa_i} \mathbb{R}^d$$
$$[x_0 : \dots : x_d] \longrightarrow \left(\frac{x_0}{x_i}, \dots \frac{\hat{x}_i}{x_i}, \frac{x_d}{x_i} \right)$$

$$\mathbb{R}^{d+1} - \{0\} \xrightarrow{f} Y \quad \text{(universal property of quot. topology)}$$
$$\mathbb{R}^{d+1} - \{0\} \xrightarrow{\pi} \mathbb{RP}^d$$
$$\mathbb{RP}^d \xrightarrow{\bar{f}} Y$$
$$\bar{f} \text{ continuous} \Leftrightarrow f \text{ continuous}$$

κ_i is continuous because $\kappa_i \circ \pi|_{\pi^{-1}(U_i)} : x \longrightarrow \left(\frac{x_0}{x_i}, \dots \frac{\hat{x}_i}{x_i}, \frac{x_d}{x_i}\right)$ is continuous (we use here the universal property of the quotient topology).
The coordinate changes are rational functions defined on hyperplane complements and therfore smooth ($= C^\infty$), for instance

$$\{y \in \mathbb{R}^d / y_1 \neq 0\} \xrightarrow{\kappa_0^{-1}} U_0 \cap U_1 \xrightarrow{\kappa_1} \{y \in \mathbb{R}^d / y_1 \neq 0\}$$
$$(y_1, \dots y_d) \longrightarrow [1 : y_1 : \dots : y_d] \longrightarrow \left(\frac{1}{y_1}, \frac{y_2}{y_1}, \dots \frac{y_d}{y_1}\right)$$

We see that the atlas is C^∞-differentiable and defines a C^∞-differentiable (smooth) structure on $\mathbb{RP}^d$.

The differentiable structure enables us to transfer local analytic concepts from eucl. spaces to local eucl. spaces.
Let $X^d \xrightarrow{F} Y^{d'}$ be a continuous map equipped with $C^{k \geq 1}$-differentiable structures. For $x \in X$ and charts $X \supset U \xrightarrow{\kappa} \kappa(U)$ of X around x and $Y \supset U' \xrightarrow{\kappa'} (U')$ of Y around $F(x)$, we call

$$\underbrace{\kappa(\underbrace{U \cap F^{-1}(U')}_{\text{open in X}})}_{\text{open nbh. of } \kappa(x) \text{ in } \mathbb{R}^d} \xrightarrow{\kappa' \circ F \circ \kappa^{-1}} \kappa'(U') \subset \mathbb{R}^{d'}$$

Definition 1.11. F is called l times resp. C^l-differentiable in x, if the local coordinate representations $\kappa' \circ F \circ \kappa^{-1}$ of F near x are l times, resp. C^l-differentiable in $\kappa(x)$.

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Definition 1.12. Let X, Y be $\mathscr{C}^k$–manifolds. A function $F\colon X \to Y$ is called l *times differentiable ($\mathscr{C}^l$-differentiable)* in x ($1 \leq l \leq k$) if there exist local coordinates κ around x and κ' around $F(x)$ such that $\kappa' \circ F \circ \kappa^{-1}$ is l times differentiable at $\kappa(x)$.

Remark.

1. A function $G\colon \mathbb{R}^d \supset V \to \mathbb{R}^{d'}$ is l times differentiable in $y \in V$ if G is $\mathscr{C}^{l-1}$-differentiable on an open neighbourhood of y in V and the $(l-1)$-st differential is differentiable in y.

2. Differentiablility does not depend on the choice of local coordinates: Let κ_1, κ_2 be two coordinate systems near $x \in X$ and κ_1', κ_2' coordinate systems near $F(x) \in Y$, then
$$\kappa_2' \circ F \circ \kappa_2^{-1} = (\kappa_2' \circ \kappa_1'^{-1}) \circ (\kappa_1' \circ F \circ \kappa_1^{-1}) \circ (\kappa_1 \circ \kappa_2^{-1})$$

and $(\kappa_2' \circ \kappa_1'^{-1})$ as well as $(\kappa_1 \circ \kappa_2^{-1})$ are $\mathscr{C}^k$-diffeomorphisms, so $(\kappa_2' \circ \kappa_2^{-1})$ is $\mathscr{C}^l$-differentiable if and only if $(\kappa_1' \circ F \circ \kappa_1^{-1})$ is $\mathscr{C}^l$-differentiable.

3. If $F\colon X \to \mathbb{R}^{d'}$ is a continuous function, then we regard $\mathbb{R}^{d'}$ as a locally Euclidean space equipped with the natural differentiable structure generated by the atlas $\{\mathrm{id}_{\mathbb{R}^{d'}}\}$, and define differentiability using the local coordinate representations $F \circ \kappa^{-1}$.

We denote the space of all $\mathscr{C}^l$-maps $X \to Y$ with $\mathscr{C}^l(X,Y)$. In particular $C^l(X) := \mathscr{C}^l(X,\mathbb{R})$. The composition of $\mathscr{C}^l$-maps is $\mathscr{C}^l$.

Definition 1.13. A homeomorphism $F\colon X \to Y$ of $\mathscr{C}^k$–manifolds X, Y is called a $\mathscr{C}^l$-*diffeomorphism*, $1 \le l \le k$, if F and F^{-1} are $\mathscr{C}^l$-differentiable. A function $F\colon X \to Y$ is a *local $\mathscr{C}^l$-diffeomorphism* if every point in X has an open neighbourhood U such that $F|_U\colon U \to F(U)$ is a $\mathscr{C}^l$-diffeomorphism onto an open subset $F(U) \subset Y$.

Example.

1. The natural 2:1 covering $S^d \to \mathbb{R}P^d$ is a local $\mathscr{C}^\infty$-diffeomorphism.
2. For any $A \in \mathrm{O}(d+1)$, $A\colon S^d \to S^d$ is a $\mathscr{C}^\infty$-diffeomorphism, and for all $A \in \mathrm{GL}(d+1,\mathbb{R})$, $A\colon \mathbb{R}P^d \to \mathbb{R}P^d$ is a $\mathscr{C}^\infty$-diffeomorphism.

Definition 1.14.

1. A *topological manifold ($\mathscr{C}^0$-manifold)* is a locally Euclidean Hausdorff space whose topology admits a countable basis (is 2nd countable).
2. A *$\mathscr{C}^k$-differentiable manifold* is a topological manifold together with a $\mathscr{C}^k$-differentiable structure.

Remark. One asks the Hausdorff property to be able to seperate points by continuous functions, and the 2nd axiom of countablility implies the existence of a partition of unity.

Example.

1. The Euclidean space $\mathbb{R}^n$ is locally Euclidean, Hausdorff and 2nd countable. Its $\mathscr{C}^\infty$-differentiable structure is generated by the atlas $\{\mathrm{id}_{\mathbb{R}^n}\}$.
2. The $\mathscr{C}^k$-submanifolds of $\mathbb{R}^n$ are in a natural way $\mathscr{C}^k$-differentiable manifolds.
3. $\mathbb{R}P^d$ is a $\mathscr{C}^\infty$-differentiable manifold, because the Hausdorff and 2nd countability properties carry over from $\mathbb{R}^{d+1}$.

Definition 1.15. Let M be an m-dimensional $\mathscr{C}^{k\ge 0}$-differentiable manifold. A subset $N \subset M$ is an n-dimensional *$\mathscr{C}^{l\le k}$-submanifold*, if around any point $x \in N$ there exists a $\mathscr{C}^l$-differentiable chart (U,κ) such that $\kappa(N \cap U) = \mathbb{R}^n \times \{0\}^{m-n} \cap \kappa(U)$. Then N is (with respect to the relative topology inherited from M) n-dimensional locally Euclidean. The Hausdorff and 2nd countability properties carry over from M to N. The restrictions of the charts generate the natural differentiable structure of N. So submanifolds are manifolds in a natural way.

Example.

1. Consider $(\mathbb{R} \times \{0\} \cup \mathbb{R} \times \{1\})/\sim$ where $(x,0) \sim (x,1)$ for all $x \in \mathbb{R} \setminus \{0\}$. This space is 2nd countable and locally Euclidean, but not Hausdorff.

2. The *long line*: Let $(W,<)$ be a well-ordered set and $L = W \times [0,1)$ and give it the lexicographical order, i.e.

$$(w,t) < (w',t') \iff w < w' \vee (w = w' \wedge t < t').$$

This order induces an order topology (generated by the the open intervals). We call $x \in W$ good, if $L_{(x,0)} := \{(w,t) < (x,0)\}$ is empty or homeomorphic to $[0,1)$. We observe the following:

   a) A limit of an increasing sequence of good elements of W is also good.

   b) if $W_x = \{w \mid w < x\}$ is countable, x is good. To prove this, assume x is bad. Without loss of generality all $w \in W_x$ are good. (If not, replace x by the smallest bad w in W_x). There exists a sequence $(x_n) \subset W_x$ such that $x_n \nearrow x$, so x is good.

   c) If W_x is uncountable, x is bad, because $L_{(x,0)}$ contains uncountably many disjoint open subsets.

To construct the long line, choose W uncountable such that all initial segments W_x are countable. Then for $L^+ = L \setminus \min L$ holds:

   a) L^+ is 1-dimensional locally Euclidean, since it is covered by $L_{(x,0)} \cong [0,1)$.

   b) L^+ is path connected.

   c) The topology of L^+ has no countable basis, because there exists an uncountable family of disjoint open sets.

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Remark. One can equip a set directly with a structure as differentiable manifold by providing a suitable atlas. The topology will result implicitly. We start with the following data: A set M and an atlas consisting of charts

$$\kappa_i \colon U_i \to V_i \quad i \in I$$

which are bijections from $U_i \subset M$ to open subsets $V_i \subset \mathbb{R}^d$, such that the coordinate changes $\kappa_j \circ \kappa_i^{-1}$ are defined on open subsets and $\mathscr{C}^k$-differentiable.

The topology arises as follows. We define the neighbourhoods of a point p on the subset $W \subset M$ such that for the charts κ_i with $pinU_i$ the subset $\kappa_i(W \cap U_i)$ is a neighbourhood of $\kappa_i(p)$ in V_i. This is independent of the chart κ_i! The axioms for neighbourhoods are satisfied (finite intersections and larger subsets are again neighbourhoods).

The open subsets are then those which are neighbourhoods of all their points. We obtain:

$$O \subset M \Leftrightarrow \kappa_i(O \cap U_i) \subset V_i \text{ open} \quad \forall \kappa_i$$

With this topology on M the charts κ_i become homeomorphisms of open subsets and M become a d-dimensional locally Euclidean space. If the atlas is countable (or has a countable subatlas), the topology of M has a countable basis. The Hausdorff property does in general not follow for free and has to be verified in the concrete case at hand. The given atlas yields a $\mathscr{C}^k$-differentiable structure on M.

Example ($\mathbb{R}P^d$ revisited)**.** On the set $\mathbb{R}P^d$ of 1-dimensional subspaces of $\mathbb{R}^{d+1}$ we consider the same atlas as before

$$\kappa_i \colon U_i = \{[x] \mid x_i \neq 0\} \to \mathbb{R}^d, [x] \mapsto \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_d}{x_i}\right)$$

The coordinate changes are defined on hyperplane complements and smooth. The Hausdorff property is clear. Hence we obtain on $\mathbb{R}P^d$ a structure as smooth manifold.

Example. The *Grassmannian* $\mathrm{Gr}_k(\mathbb{R}^n)$ is the space of all k-dimensional linear subspaces $L^k \subset \mathbb{R}^n$, $1 \leq k \leq n-1$. We will put on it a structure as a smooth manifold by giving a suitable atlas. Every direct sum decomposition $\mathbb{R}^n = V^k \oplus W^{n-k}$ yields a chart

$$\kappa_{V,W} \colon U_{V,W} = \{L^k \subset \mathbb{R}^n \mid L \pitchfork W\} \to \mathrm{Hom}(V,W),\ \mathrm{graph}(F) \mapsto F$$

We investigate the domains of definition and smoothness of the coordinate changes. The components with respect to two decompositions $V \oplus W = \mathbb{R}^n = V' \oplus W'$ transform into each other linearly, i.e.

$$v + w = v' + w' \Rightarrow \begin{cases} v' = Av + Bw \\ w' = Cv + Dw \end{cases}$$

where A, B, C, D are linear maps such that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is a block decomposition of $\mathrm{id}_{\mathbb{R}^n}$ with respect to the two direct sum decompositions. Let $L \in U_{V,W} \cap U_{V',W'}$, i.e. $L \pitchfork W, W'$, and let $\mathrm{graph}(F) = L = \mathrm{graph}(F')$ with $F \in \mathrm{Hom}(V,W)$ and $F' \in \mathrm{Hom}(V',W')$. We decompose the vectors in L and obtain

$$v + Fv = v' + Fv' \Rightarrow \begin{cases} v' = Av + BFv = (A+BF)v \\ F'v' = Cv + DFv = (C+DF)v \end{cases}$$

where $A + BF \in \mathrm{Hom}(V,V')$ is invertible, because $L \pitchfork W, W'$, so

$$F' = (C+DF)(A+BF)^{-1}$$

Hence the coordinate changes are defined on open subsets and are smooth.

There exist finite subatlasses (e.g. corresponding to decompositions by coordinate subspaces), and hence a countable basis for the topology of $\mathrm{Gr}_k(\mathbb{R}^n)$. The Hausdorff property is easy to check (exercise). We obtain on $\mathrm{Gr}_k(\mathbb{R}^n)$ a stucture as a smooth manifold.

Remark.

1. The group $\mathrm{GL}(n,\mathbb{R})$ operates on $\mathrm{Gr}_k(\mathbb{R}^n)$ by diffeomorphisms, because it preserves our atlas.
2. The projective space is a special case of the Grassmannian, $\mathrm{Gr}_1(\mathbb{R}^n) = \mathbb{R}P^{n-1}$, and $\mathrm{Gr}_{n-k}(\mathbb{R}^n)$ is diffeomorphic to $\mathrm{Gr}_k(\mathbb{R}^n)$.

1.4 The tangent bundle of a differentiable manifold

1.4.1 Tangent spaces to submanifolds of Euclidean space

Let first $A \subset \mathbb{R}^n$ be an arbitrary subset. A vector $v \in \mathbb{R}^n$ is a *tangent vector* to A at $p \in A$ if it is the velocity vector of a curve in A through p, i.e. there is a $\mathscr{C}^1$-differentiable curve

$$c\colon (-\varepsilon, \varepsilon) \to A$$

such that $c(0) = p$ and $\dot{c}(0) = v$.

The set T_pA of all tangent vectors to A in p may be called the *tangent (double) cone* to A in p, since

$$v \in T_pA \Rightarrow \mathbb{R}v \subset T_pA$$

because

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} c(\lambda t) = \lambda \dot{c}(0)$$

The tangent cone is preserved by (differentials of) diffeomorphisms: If $A_1, A_2 \subset \mathbb{R}^n$ are subsets and if $\psi U_1 \to U_2$ is a diffeomorphism of open subsets such that $\psi(A_1 \cap U_1) = A_2 \cap U_2$, then for every point $p_1 \in A_1 \cap U_1$ and $p_2 = \psi(p_1) \in A_2 \cap U_2$ holds

$$\mathrm{d}\psi_{p_1}(T_{p_1}A_1) = T_{p_2}A_2.$$

Indeed, if $c_1\colon (-\varepsilon, \varepsilon) \to A_1 \cap U_1$ is a differentiable curve then $c_2 := \psi \circ c_1$ is a differentiable curve in $A_2 \cap U_2$ and the chain rule implies

$$\mathrm{d}\psi(\dot{c}_1(0)) = (\psi \circ c_1)'(0) = \dot{c}_2(0).$$

For linear subspaces $L \subset \mathbb{R}^n$ holds

$$T_xL = L \quad \forall x \in L$$

It follows for submanifolds: The tangent cones T_pM to a d-dimensional $\mathscr{C}^{k\geq 1}$-differentiable submanifold $M \subset \mathbb{R}^n$ are d-dimensional linear subspaces and are called the *tangent spaces* to M at $p \in M$.

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The property of T_pM of being a linear subspace can also be seen as follows: If $V' \xrightarrow{F} M \cap U$ is a local C^k-parametrization near p, $F(0) = p$, and if $U \xrightarrow{s} \mathbb{R}^{n-d}$ is a C^k-submersion s.t. $M \cap U = s^{-1}(0)$, then we have

$$\underbrace{\mathrm{im} dF_0}_{\dim = d} \subseteq T_pM \subseteq \underbrace{\ker \underbrace{dS_p}_{surj.}}_{\dim = d}$$

$\Rightarrow$ equality holds: T_pM is d-dim linear subspace.
Towards an intrinsic definition of the tangent space:
Two curves $(-\epsilon_i, \epsilon_i) \xrightarrow[C^1]{c_i} M$ represent the same tangent vector if they agree up to first order in 0, i.e.

$$c_1(0) = c_2(0) \quad \wedge \quad \dot{c}_1(0) = \dot{c}_2(0)$$

This equivalence relation on differentiable curves in M can be expressed in terms of local coordinate charts and one thereby obtains an intrinsic definition of tangent spaces which carries over to…

1.4.2 Tangent vectors and differentials

Let M^m be a $C^{k\geq 1}$ manifold. We say that two C^1-curves $(-\epsilon_i, \epsilon_i) \xrightarrow{c_i} M$ agree up to first order in 0, if $c_1(0) = c_2(0) := p$ and if for a chart around p holds

$$(\kappa \circ c_1)^{\dot{}}(0) = (\kappa \circ c_2)^{\dot{}}(0).$$

This is independent of the chart:

$$\underbrace{(\kappa' \circ c_i)^{\dot{}}(0)}_{\text{coo. change } (\kappa'\circ\kappa^{-1})\circ(\kappa\circ c_i)} \underbrace{=}_{\text{chain rule}} \underbrace{d(\kappa' \circ \kappa^{-1})_{\kappa(p)}}_{\text{invertible}} \Big((\kappa \circ c_i)^{\dot{}}(0)\Big)$$

An equivalence class of curves agreeing up to first ofder is called a 1-jet (of curves) (here in p).

Definition 1.16. A tangent vector to M is a 1-jet.
The set of all tangent vectors to M in a point $p \in M$ is called the tangent space T_pM.
The set $TM = \bigcup_{p\in M} T_pM$ of all tangent vectors is called the tangent bundle.

Remark: Tangent spaces to different points are disjoint (think of them as vertical to the manifold).
A differentiable map $M^m \xrightarrow{F} N^n$ of differentialbe mainfolds induces the map of tangent bundles $TM \xrightarrow{dF} TN, [c] \longrightarrow [F \circ c]$, the differential of F. It is well defined because for charts κ around $c(0)$ and κ' around $F(c(0))$ holds:

$$\kappa' \circ (F \circ c) = \underbrace{(\kappa' \circ F \circ \kappa^{-1})}_{\text{loc. coo. rep. of F}} \circ (\kappa \circ c)$$

$$\Rightarrow \text{chain rule:} \quad (\kappa' \circ (F \circ c))^{\dot{}}(0) = \underbrace{d(\kappa' \circ F \circ \kappa^{-1})_{\kappa(c(0))}}_{\text{old differential (Ana II)}} \Big((\kappa \circ c)^{\dot{}}(0)\Big).$$

Hence $[F \circ c]$ depends only on $[c]$.
The differential maps tangent spaces into tangent spaces, $dF(T_pM) \subseteq T_{F(p)}N$ and we write $dF_p := dF|_{T_pM}$. The chain rule holds: If $M \xrightarrow{F} N \xrightarrow{G} W$ are diff. manifolds, then $d(G \circ F) = dG \circ dF$ because

$$d(G \circ F)([c]) = [G \circ F \circ c] = dG[F \circ c] = dG(dF([c])).$$

Remark: The differentiable structure as defined above is consistent with the "old" differential because for a diff. map

$$\mathbb{R}^m \supset U \text{ open} \xrightarrow{F} \mathbb{R}^n, \quad (F \circ c)^{\dot{}}\,(0) = dF_{c(0)} \cdot \dot{c}(0)$$

holds according to the old chain rule.

1.4.3 The linear structure on tangent spaces

For open subsets $U \subset \mathbb{R}^n$, we have the natural identifications

$$T_pU \xrightarrow[c \longrightarrow \dot{c}(0)]{\cong} \mathbb{R}^n,$$

and therefore linear structures on the tangent spaces. The differentials of differentiable maps $U \longrightarrow \mathbb{R}^m$ are linear. This carries over (via charts) to abstract manifolds.
Let M_m be a differentiable manifold. A chart (U, κ) around $p \in M$ yields the identification

$$T_pM \xrightarrow[1:1]{d\kappa_p} T_{\kappa(p)}U \cong \mathbb{R}^m$$

And thus a linear structure on T_pM. It does not depend on the chart because according to the chain rule

$$d\kappa'_p \circ \underbrace{(d\kappa_p)^{-1}}_{(d\kappa^{-1})_{\kappa p}} = \underbrace{d(\kappa' \circ \kappa^{-1})_{\kappa(p)}}_{\text{old differential, hence linear}}$$

is an isomorphism of vector spaces. It is clear that the differentials act linearly on tangent spaces i.e. that the maps

$$dF_p : T_pM \longrightarrow T_{F(p)}N$$

are linear.
Local coordinates distinguish bases of tangent spaces: Let (U,x) be a local chart around p. W.r.t. the identification $(*)$ $T_pM \longrightarrow \mathbb{R}^m, [c] \longrightarrow (x \circ c)^{\dot{}}\,(0)$ the standard basis vector $e_i \in \mathbb{R}^m$ corresponds to the tangent vector $[c_i]$ with $c_i(t) = x^{-1}(x(p) + te_i)$ near $t = 0$.
For reasons which become clear later (when we interpret tangent vectors as directoral derivatives) we use the notation

$$\frac{\partial}{\partial x_i}|_p := [c_i]$$

Definition 1.17. $\{\frac{\partial}{\partial x_i}|_p\}$ is called the standard basis of T_pM w.r.t. local coordinates x. An arbitrary tangent vector $[c] \in T_pM$ corresponds via $(*)$ to the vector $(x \circ c)^{\dot{}}\,(0) = \sum i(x_i \circ c)^{\dot{}}\,(0)e_i$, and we obtain the representation $[c] = \sum_{i=1}^{m} (x_i \circ c)^{\dot{}}\,(0)\frac{\partial}{\partial x_i|_p}$

Matrix representation of the differential $dF : TM \longrightarrow TN$ w.r.t. the standard basis:

$$\begin{array}{ccc} M^m & \xrightarrow{F} & N^n \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle y} \\ V_M \subset \mathbb{R}^m & \xrightarrow[=y \circ F \circ x^{-1}]{\tilde{F}} & V_N \subset \mathbb{R}^n \end{array} \qquad \begin{array}{ccc} [c] & \xrightarrow{dF_p} & [F \circ c] \\ {\scriptstyle (*)}\downarrow & & \downarrow{\scriptstyle (*)} \\ (x \circ c)^{\dot{}}(0) & \xrightarrow[d\tilde{F}_{x(p)}]{} & (y \circ F \circ c)^{\dot{}}(0) \end{array}$$

W.r.t the standard bases $\{e_j\}$ resp. $\{e_i\}$ of $\mathbb{R}^m$ and $\mathbb{R}^n$, $d\tilde{F}_{x(p)}$ is given by the Jakobi matrix $\left(\frac{\partial \tilde{F}_i}{\partial x_j}(x(p))\right)_{i=1,\dots n}$. These correspond via the differentials of local charts x resp. y to the associated standard bases $\{\frac{\partial}{\partial x_j}|_p\}$ of T_pM, resp. $\{\frac{\partial}{\partial x_i}|_{F(p)}\}$ of $T_{F(p)}N$.
Therefore

$$dF_p \frac{\partial}{\partial x_j}|_p = \sum_{i=1}^{n} \frac{\partial \tilde{F}_i}{\partial x_j}(x(p)) \frac{\partial}{\partial y_i}|_{F(p)}$$

Transformation of standard bases in case of coordinate change: We put M = N and $F = \mathrm{id}\ M$. We write the coordinate change suggestively as $\tilde{x}(x)$. Then

$$\frac{\partial}{\partial x_j}|_p = \sum_{i=1}^{n} \frac{\partial \tilde{x}_i}{\partial x_j}(x(p)) \frac{\partial}{\partial \tilde{x}_i}|_p,$$

respectively,

$$\frac{\partial}{\partial \tilde{x}_i}|_p = \sum_{j=1}^{m} \frac{\partial x_j}{\partial \tilde{x}_i}(\tilde{x}(p)) \frac{\partial}{\partial \tilde{x}_j}|_p,$$

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1.4.4 The differentiable structure on the tangent bundle

LetM^m be a $\mathscr{C}^k$–manifold. The natural projection

$$TM = \coprod_{p\in M} T_pM \to M$$

is given by the footpoint projection, i.e. $\pi^{-1}(p) = T_pM$. A chart (U, x) of M induces local coordinates:

$$Tx\colon TU = \coprod_{p\in U} T_pU \to x(U) \times \mathbb{R}^m \subset \mathbb{R}^{2m}, \qquad \sum_i v_i \frac{\partial}{\partial x_i}\Big|_p \mapsto \left(x(p), \sum_i v_i e_i\right)$$

which we will use as a chart for TM. If $(\widetilde{U}, \widetilde{x})$ is another chart on M, then the coordinate change is given by

$$x(U \cap \widetilde{U}) \times \mathbb{R}^m \to \widetilde{x}(U \cap \widetilde{U}) \times \mathbb{R}^m, \qquad (x(p), v) \mapsto \left(\widetilde{x}(p), \sum_i \sum_j v_j \frac{\partial \widetilde{x}_i}{\partial x_j}(x(p))\widetilde{e}_i\right).$$

We see that coordinate changes are defined an open sets and are $\mathscr{C}^{k-1}$–differentiable. It follows (compare the technical remark above) that the atlas consisting of the charts Tx yields on the set TM a natural topology as $2m$–dimensional locally Euclidean space and a $\mathscr{C}^{k-1}$–differentiable structure. The topology on TM has a countable basis since there are countable subatlases. The Hausdorff property is clear. We conclude

Theorem 1.18. *If M is an m–dimensional $\mathscr{C}^{k\geq 1}$–manifold, then TM carries a natural induced structure as a $2m$–dimensional $\mathscr{C}^{k-1}$–manifold.*

Remark. The projection $\pi\colon TM \to M$ is $\mathscr{C}^{k-1}$–differentiable, which is the maximum possible degree of differentiability.

If $F\colon M \to N$ is a $\mathscr{C}^l$–map of $\mathscr{C}^k$–manifolds with $1 \le l \le k$, then a local coordinate representation shows that its differential $\mathrm{d}F\colon TM \to TN$ is $\mathscr{C}^{l-1}$–differentiable.

$$\mathrm{d}F_p\left(\left.\frac{\partial}{\partial x_j}\right|_p\right) = \sum_i \frac{\partial \widetilde{F_i}}{\partial x_j}(x(p))\left.\frac{\partial}{\partial y_i}\right|_{F(p)} \qquad \widetilde{F} = y \circ F \circ x^{-1}$$

because the entries of the Jacobian matrix depend $\mathscr{C}^{l-1}$–differentiably on the point.

1.4.5 Tangent vectors as derivations

One can regard tangent vectors (analytically) as differential operators. This will be useful for us from a technical point of view. To a tangent vector $[c] \in T_pM$ we can assign its *directional derivative*

$$\partial_{[c]}f \coloneqq \mathrm{d}f[c] = [f \circ c] = (f \circ c)'(0)$$

where we canoncially identify $T_{(f\circ c)(0)}\mathbb{R}$ with $\mathbb{R}$. The differential operator $\partial_{[c]}$ has the following properties:

1. It is $\mathbb{R}$–linear:
$$\partial_{[c]}(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \partial_{[c]} f_1 + \lambda_2 \partial_{[c]} f_2$$

2. It is a *derivation*, i.e. it satisfies the product rule
$$\partial_{[c]}(fg) = (\partial_{[c]}f)g(p) + f(p)(\partial_{[c]}f)$$

Directional derivatives are local operatores, i.e. $\partial_{[c]}f$ depends only on the values of f near p. It is therefore natural to pass to germs of functions. Let M be a $\mathscr{C}^{k\ge 1}$-manifold and $p \in M$. We call two functions defined on neighbourhoods U_1 resp. U_2 of p *equivalent*, if they agree on a neighbourhood $W \subset U_1 \cap U_2$ of p. An equivalence class is called a *germ* (of a function) in p, and $\mathscr{C}^k$–differentiable, if the functions representing it are $\mathscr{C}^k$–differentiable in a neighbourhood of p. We denote the germ of f in p by $[f]_p$. The set $\mathscr{C}^k(M)_p$ of $\mathscr{C}^k$–germs in p carries a natural structure as an $\mathbb{R}$–algebra.

Definition 1.19. A *derivation* on M in p is an $\mathbb{R}$–linear functional $D\colon \mathscr{C}^k(M)_p \to \mathbb{R}$ which satisfies the product rule:

$$D(fg) = (Df)g(p) + f(p)(Dg)$$

The derivations in p form an $\mathbb{R}$–vector space $\mathscr{D}_p(M)$. It contains the tangent space, i.e. there is the natural linear embedding

$$T_pM \to \mathscr{D}_p(M), v \mapsto \partial_v \tag{$*$}$$

If (U,x) are local coordinates at p, thento the tangent vector $\frac{\partial}{\partial x_i}|_p$ is assigned the derivation

$$f \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(f\circ x^{-1})(x(p)+te_i) = \frac{\partial(f\circ x^{-1})}{\partial x_i}(x(p))$$

which is the i–th partial derivative wrt. the coordinates x. This motivates our notation $\frac{\partial}{\partial x_i}\big|_p$. From now on we identify the tangent vector v with the corresponding derivation ∂_v. Let us discuss the surjectivity of $(*)$. We denote by $I_p \subset \mathscr{C}^k(M)_p$ the maximal ideal of the germs vanishing in p, i.e. the krenel of the evaluation map (algebra homomorphism!) $\mathscr{C}^k(M)_p \to \mathbb{R}$. We observe that every derivation $D\colon \mathscr{C}^k(M)_p \to \mathbb{R}$ vanishes on germs of constant functions, since

$$D(1) = D(1)\cdot 1 + 1\cdot D(1) - D(1) = D(1\cdot 1) - D(1) = D(1) - D(1) = 0$$

and as well on the idal I_p^2 (the linear combinations of products $f_1 f_2$ for $f_1 f_2 \in I_p$), because

$$D(f_1 f_2) = D(f_1)f_2(p) + f_1(p)D(f_2) = D(f_1)\cdot 0 + 0\cdot D(f_2) = 0 \qquad f_1, f_2 \in I_p$$

Vice versa, every linear functional $D\colon \mathscr{C}^k(M)_p \to \mathbb{R}$, which vanishes on $\mathbb{R}\cdot 1 \cup I_p^2 \subset \mathscr{C}^k(M)$ is a derivation:

$$\begin{aligned}
D((c_1\cdot 1 + f_1)(c_2\cdot 1 + f_2)) &= D(c_1c_2\cdot 1 + c_2 f_1 + c_1 f_2 + f_1 f_2)\\
&= c_2 D(f_1) + c_1 D(f_2)\\
&= D(c_1\dot{1} + f_1)(c_2\cdot 1 + f_2)(p) + (c_1\dot{1} + f_1)(p)\cdot D(c_2\cdot 1 + f_2)
\end{aligned}$$

Hence derivations correspond to linear forms on the vector space I_p/I_p^2:

$$\mathscr{D}_p(M) \cong (I_p/I_p^2)^*$$

Proposition 1.20. *If $k=\infty$, then the embedding $(*)$ is a linear isomorphism.*

Proof. Let (U,κ) be a chart around p with $\kappa(p)=0$ and $[f]\in I_p$. Then we obtain for x near 0:

$$(f\circ\kappa^{-1})(x) = \int_0^1 \left(\frac{\mathrm{d}}{\mathrm{d}t}(f\circ\kappa^{-1})(tx)\right)\mathrm{d}t = \sum_i x_i \underbrace{\int_0^1 \frac{\partial(f\circ\kappa^{-1})}{\partial x_i}(tx)\,\mathrm{d}t}_{=:g_i(x)}$$

so

$$f = \sum_{i=1}^{\dim M}(x_i\circ\kappa)(g_i\circ\kappa) \quad\Rightarrow\quad [f] = \sum_{i=1}^{\dim M}[x_i\circ\kappa]g_i(0) \mod I_p^2.$$

Therefore I_p/I_p^2 is generated by the $[x_i\circ\kappa] + I_p^2$ and $(*)$ is also surjective by dimension reasons. □

Remark. If $1\le k<\infty$, then $\dim\mathscr{D}_p(M) = \dim I_p/I_p^2 = \infty$, because the germs in I_p^2 are even $(k+1)$–times differentiable.

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1.4.6 Submersions and Immersions

After having defined the differential, we can generalize these notions to the setting of manifolds.
Let $M^m \xrightarrow{F} N^n$ be a $C^{k\geq 1}$-map of C^k-manifolds.

Definition 1.21. F is called a <u>submersion</u> if its differential dF_p is surjective at all points $p \in M$.

We know (compare our discussion of submanifolds and the Inverse Function Theorem):

Proposition 1.22. *The inverse images $F^{-1}(y)$ of values of submersions are submanifolds.*

Definition 1.23. i) F is called an <u>immersion</u>, if dF_p is injective for all $p \in M$.

ii) F is called <u>embedding</u>, if its image $F(M) \subset N$ is a submanifold and $M \xrightarrow{F} F(M)$ is a diffeomorphism.

immersions are locally embeddings (compare the earlier discussion of local parametrizations of submanifolds)

Proposition 1.24. *Every point $p \in M$ has an open neighbourhood U s.t. $F|_U$ is an embedding*

Proof. We thicken F locally to make it a diffeomorphism (working in a single chart). For a sufficiently small open nbh. U of p (contained in one chart) and $\epsilon > 0$ exists an extension of $F|_U$ to $U \times (-\epsilon, \epsilon)^{(n-m)} \xrightarrow[C^k]{\overline{F|_U}} N$ s.t. $(d\overline{F|_U})_{(p,0)}$ is invertible.
According to Inverse Function theorem we can achieve by shrinking U and ϵ, that $F|_U$ is a diffeomorphism. then $F|_U$ is an embedding. □

injective immersions are in general no embeddings!

Proposition 1.25. *An immersion is an embedding if and only if it is a homeomorphism onto its image.*

Proof. " $\Leftarrow$ ": Let $p \in M$. by the previous proposition, there exists an open neighbourhood U of p s.t. $F(U) \subset N$ is a submanifold. By assumption, $F(U)$ is open in $F(M)$, hence $\exists O \in N$ open s.t. $F(U) = F(M) \cap O$.
It follows that $F(M)$ is a submanifold near $F(p)$. Thus $F(M) \subset N$ is a submanifold. Then

$$M \xrightarrow[homeo+imm]{F} \underbrace{F(M)}_{(sub)manifold}$$

is a local diffeomorphism. Since it is also a homeomorphism, it is also a global manifold. □

Corollary 1.26. *If M is compact, then every injective immersion is an embedding.*

Proof. Bijective continuous maps from compact spaces to Hausdorff spaces are homeomorphisms. □

We call the images of injective immersions $M \to N$ immersed submanifolds of N (Relaxing our notion of (embedded) submanifolds.)

1.5 Vector fields, flows, Lie brackets

(everything C^∞)

Definition 1.27. A (smooth) vector field on M is a smooth map $X : M \to TM$ with $\pi \circ M = \mathrm{id}_M$ i.e. $X(p) \in T_pM\ \forall p \in M$.

Geometric intuition: In every point, a direction is chosen.
in local coordinates (U,x) the vector field can be expressed in terms of the cononical basis:

$$X(p) = \sum a_i(p) \frac{\partial}{\partial x_i}|_p.$$

The smoothness of X on U is equivalent to the smoothness of the coefficient functions a_i.
Analytically, we can regard vector fields as certain differential operators on functions. To the vectorfield X corresponds the operator, also denoted by X:

$$\begin{aligned} X : C^\infty(M) &\longrightarrow C^\infty(M) \\ f &\longrightarrow Xf : p \longrightarrow \underbrace{X(p)}_{derivation}\ f. \end{aligned}$$

"derivative in direction of the vector field X"
Properties:

i) $\mathbb{R}$- linear

ii) product rule: $X(fg) = (Xf)g + f(Xg)$

Vice versa, to an $\mathbb{R}$ -linear operator X satisfying the product rule corresponds a vector field. Namely, X determines in a point $p \in M$ the derivation $X(p) \in T_pM$ given by $f \xrightarrow{X(p)} (X(f))(p)$ for $f \in C^\infty(M)$.
We use:

Lemma 1.28. *This operator is local, i.e. f := 0 near p* $\Rightarrow (Xf)(p) = 0$

Proof. —Unvollständig———

$$\Rightarrow 0 = X(\varphi f) = (X\varphi) \underbrace{f}_{:=0} + \underbrace{\varphi}_{:=1 \text{ near p}} (Xf) \Rightarrow Xf := 0$$

near p, i.p. (Xf)(p) = 0. □

The smoothness of the vector field $p \to X(p)$ determined by the operator X can be seen near a point p by choosing local coordinates:

$$X(q) = \sum_{i=1}^{m} (Xh_i)(q) \frac{\partial}{\partial x_i}|_q \qquad \text{for q near p,}$$

where the h_i are auxiliary functions which coincide near p with the i.th coordinate function.
The space $\Gamma(TM)$ of smooth vector fields on M is a module over the ring $C^\infty(M)$, where multiplication is defined pointwise:

$$(\varphi X)_{(p)} = \varphi(p)X(p) \quad \text{for } \varphi \in C^\infty(m), X \in \Gamma(TM).$$

In terms of derivations:

$$(\varphi X)f = \varphi(Xf), \quad \varphi, f \in C^\infty(M), X \in \Gamma(TM).$$

An <u>integral curve</u> or <u>trajectory</u> of the vector field X is a differential curve c: $I \to M$ with

$$\dot{c} = X \circ c \quad (*)$$

where

$$\dot{c} = [\tau \to c(t+\tau)] = dc \underbrace{[\tau \to t+\tau]}_{\frac{\partial}{\partial \tau}|_t} = dc \frac{\partial}{\partial \tau}|_t \in T_{c(t)}M.$$

We rewrite $(*)$ in local coordinates:

$$X(p) = \sum a_i(p) \frac{\partial}{\partial x_i}|_p$$

$$x \circ c = (c_1, \dots c_m); \quad \dot{c}(t) = \sum \dot{c}_i(t) \frac{\partial}{\partial x_i}|_c(t)$$

$(*)$ becomes a system of first order ODEs:

$$\dot{c}_i = a_i \circ c = \underbrace{(a_i \circ x^{-1})}_{:=\tilde{a}_i} \circ (x \circ c)$$

$\Rightarrow \dot{c}_i(t) = \tilde{a}_i(c_1(t), \dots c_m(t))$ first order smooth coefficients.
The local theory of ODEs yields that the initial value problem $\dot{c} = X \circ c, c(t_0) = p_0$ has unique local solutions, which depend smoothly on the initial conditions.
In other words: To $(p_0, t_0) \in M \times \mathbb{R}$ exists an open nbh U_0 of p_0, $\epsilon > 0$ and a smooth map $\Phi : U_0 \times (t_0 - \epsilon; t_0 + \epsilon) \longrightarrow M$

$$\text{with } \begin{cases} \frac{\partial \Phi}{\partial t} = X \circ \Phi \\ \Phi(\bullet, t_0) = \mathrm{id}_{U_0} \end{cases}, \text{where } \frac{\partial \Phi}{\partial t} := d\Phi \frac{\partial}{\partial t}.$$

The local flow $\Phi(p, \bullet)$ is the integral curve with $t_0 \to p$.
Globally one obtains: Due to the uniquenes of local solutions, there is for $p \in M$ an unique maximal integral curve

$$c_p : (\underbrace{\alpha(p)}_{\in[-\infty,0)}, \underbrace{\omega(p)}_{\in(0,+\infty]}) \longrightarrow M$$

with $c_p(0) = p$.

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The function $\alpha \colon M \to [-\infty, 0)$ is upper semicontinuous and $\omega \colon M \to (0, \infty]$ is lower semicontinuous, i.e. for $p \in M$ and $\alpha(p) < a < 0 < b < \omega(p)$ there exists a neighbourhood U of p such that c_q is defined on an interval containing $[a, b]$ for all $q \in U$. In other words, the set

$$D^X := \{(p, t) \in M \times \mathbb{R} \mid \alpha(p) < t < \omega(p)\}$$

is open in $M \times \mathbb{R}$. It is the natural domain of definition for the global flow

$$\phi^X \colon D^X \to M, (p, t) \mapsto c_p(t)$$

of X. For a vector field $X \in \Gamma(TM)$ and $p \in M$, there is a maximal integral curve $c_p \colon (\alpha(p), \omega(p)) \to M$with $c_p(0) = p$. The set D^X is open in $M \times \mathbb{R}$. The smoothness of the lcoal flows implies the uniqueness of ϕ^X. The uniqueness of the solutions to the ODE implies the group property of the flow ϕ^X:

$$\phi^X(p, t_1 + t_2) = \phi^X(\phi^X(p, t_1), t_2)$$

This holds whereever it is defined.

Definition 1.29. A vector field $X \in \Gamma(TM)$ is called *complete* if $D^X = M \times \mathbb{R}$. That is, all integral curves are defined for all $\mathbb{R}$.

In this case the maps $\phi_t := \phi^X(\_, t) \colon M \to M$ are well–defined and are diffeomorphisms. We have the group property

$$\phi_{t_1+t_2} = \phi_{t_1} \circ \phi_{t_2}$$

So the map $\mathbb{R} \to \mathrm{Diff}(M), t \mapsto \phi_t$ is a group homomorphism Such a homomorphism is called a *1–parameter–subgroup* of diffeomorphisms.

Example. In a compact manifold all vector fields are complete.

Example.

1. If there is a $\varepsilon_0 > 0$ such that all integral curves c_p of the vector field X are defined (at least) in $(-\varepsilon_0, \varepsilon_0)$, then X is complete.

2. If $\omega(p) < \infty$, then c_p leaves every compactum in M.

1.5.1 The Lie bracket

Let $X, Y \in \Gamma(TM)$ be vector fields. We think of a vector field as a derivation in order du construct the operator:

$$[X, Y] \colon \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M), f \mapsto X(Yf) - Y(Xf)$$

This is a differential operator of order ≤ 2, but because we are taking skew-symmetrization, the second order part disappears and we actually obtain an operator of first order. We verify that $[X, Y]$ is again a derivation; in particular, it is a vector field.

- $\mathbb{R}$–linearity is clear.
- The product rule can be seen by calculation:

$$\begin{aligned}[X, Y](fg) &= X(Y(fg)) - Y(X(fg)) \\ &= (XYf)g + (Yf)(Xg) + (Xf)(Yg) + f(XYg) \\ &- (YXf)g - (Xf)(Yg) - (Yf)(Xg) - f(YXg) \\ &= (XYf - YXf)g + f(XYg - YXg) = g[X, Y]f + f[X, Y]g\end{aligned}$$

Definition 1.30. The map $[\_, \_] \colon \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ is called the *Lie bracket*.

We can write the Lie bracket in local coordinates: Let $X = X^i \partial_i$, $Y = Y^i \partial_i$ and $[X, Y] = Z^i \partial_i$, then

$$\begin{aligned}[X, Y] &= X^i \partial_i (Y^j \partial_j) - Y^i \partial_i (X^j \partial_j) \\ &= X^i (\partial_i Y^j) \partial_j + X^i Y^j \partial_i \partial_j - Y^i (\partial_i X^j) \partial_j - Y^i X^j \partial_i \partial_j \\ &= X^i (\partial_i Y^j) \partial_j - Y^i (\partial_i X^j) \partial_j\end{aligned}$$

so

$$Z^j = \sum_i \left(X^i \frac{\partial Y^j}{\partial x_i} - Y^i \frac{\partial X^j}{\partial x_i} \right)$$

Remark. Our definition of the Lie bracket only works for $\mathscr{C}^\infty$ vector fields, since we are considering vector fields as derivations. We can take the description in local coordinates as a definition of the Lie bracket for $\mathscr{C}^{k \geq 1}$ vector fields. It is well defined and independent of the local coordinate chart that we chose because this is true in the $\mathscr{C}^\infty$ case. The formula also shows that the Lie bracket of two $\mathscr{C}^k$ vector fields (i.e. X^i, Y^i are $\mathscr{C}^k$ functions) is a $\mathscr{C}^{k-1}$ vector field.

Proposition 1.31 (Algebraic properties of the Lie bracket)**.** *The Lie bracket*

1. *is $\mathbb{R}$–linear.*
2. *is skew–symmetric: $[X, Y] = -[Y, X]$.*
3. *fulfills the Jacobi identity:*

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0$$

Proof. $\mathbb{R}$–linearity and skew–symmetry are clear. To prove the Jacobi identity, observe that

$$\begin{aligned}[][[X,Y],Z]f &= [X,Y]Zf - Z[X,Y]f = XYZf - YXZf - ZXYf + ZYXf\\ [[Y,Z],X]f &= [Y,Z]Xf - X[Y,Z]f = YZXf - ZYXf - XYZf + XZYf\\ [[Z,X],Y]f &= [Z,X]Yf - Y[Z,X]f = ZXYf - XZYf - YZXf + YXZf\end{aligned}$$

sums up to 0. □

Definition 1.32. An $\mathbb{R}$–vector space V together with a multiplication $[\_,\_]\colon V\times V\to V$ with the properties 1-3 is called a *Lie algebra* over $\mathbb{R}$.

The geometric meaning of the Lie bracket will become clear after we discuss the notion of a Lie derivative in the next section.

If $\varphi,\psi\in\mathscr{C}^\infty(M)$, then $[\varphi X,\psi Y]=\varphi\psi[X,Y]+\varphi(X\psi)Y-\psi(Y\varphi)X$.

1.5.2 The Lie derivative of vector fields

We cannot define a directional derivative $\partial_v Y$ of a vector field $Y\in\Gamma(TM)$ in the direction of a vector $v\in T_pM$ without some additional structure (a connection, this will be defined later) because the vectors $Y(q)\in T_qM$ live in different tanget spaces.

But with a flow ϕ^X of a vector field $X\in\Gamma(TM)$ we are able to idetify the tangent spaces along a trajectory of the the flow and in this way to derivate a vector field Y in the direction of the flow of X.

Let $X,Y\in\Gamma(TM)$. We define the *derivative of Y in the direction of X*, called *Lie derivative*, by:

$$L_XY(p)=\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}\mathrm{d}\phi^X_{-t}(Y\circ\phi^X_t(p))\in T_pM$$

The fact that this is smooth can be seen e.g. in local coordinates. Therefore the vetor field L_XY is smooth, i.e. $L_XY\in\Gamma(TM)$.

Proposition 1.33. $L_XY=[X,Y]$.

Lemma 1.34. *If $X(p)\neq 0$, then we can find local coordinates near $p\in M$ such that $X=\partial_1$.*

Proof. We choose a smooth map $q\colon B_{2\delta}(0)\subset\mathbb{R}^{n-1}\to M$ such that $q(0)=p$. It has an injective differential $\mathrm{d}q_0$ at 0 and $X(p)\notin\mathrm{Im}(\mathrm{d}q_0)$. For $\varepsilon>0$ small enough we consider the smooth function

$$H\colon(-\varepsilon,\varepsilon)\times B_\delta(0)\to M,\qquad (x_1,\dots,x_n)\mapsto\phi^X(q(x_2,\dots,x_n),x_1)$$

and see that $\mathrm{d}H_0$ is invertible ($\mathrm{d}H_0=(X(p),\mathrm{d}q_0)$). Hence H is a local diffeomorphism, i.e. for $\varepsilon,\delta>0$ small enough H is a diffeomorphism into an open neighbourhood of p. The inverse H^{-1} is a local chart with the desired property. □

Proof of proposition. We have to show that $L_X Y(p) = [X,Y](p)$ for every $p \in M$. First assume $X(p) \neq 0$. We work in local coordinates. By the lemma we can assume $X = \partial_1$. Let $\psi\colon M \supset U \to \mathbb{R}^n$ be a corresponding coordinate chart. We identify U with its image $\psi(U)$ and work in $\psi(U)$ to simplify the notation. We have $\phi_t^X(x) = x + te_1$, so $\mathrm{d}\phi_t^X = \mathrm{id}_{\mathbb{R}^n}$. From $Y \circ \phi_t^X(x) = Y(x+te_1)$ it follows that $\mathrm{d}\phi_{-t}^X(Y \circ \phi_t^X(x)) = Y(x+te_1) \in \mathbb{R}^n$. Write Y in local coordinates, $Y = \sum Y^j \partial_j|_p$, then

$$L_X Y(p) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \sum_j X^j(x+te_1)\frac{\partial}{\partial x_j}\bigg|_p = \sum_j \frac{\partial X^j}{\partial x_1}\frac{\partial}{\partial x_j}\bigg|_p = \left[\frac{\partial}{\partial x_1}, \sum_j X^j \frac{\partial}{\partial x_j}\right](p)$$

which is $[X,Y](p)$, so $L_X Y(p) = [X,Y](p)$. Now let $X(p) = 0$. Consider the set

$$B = \{q \in M \mid X(q) \neq 0\} \subset M.$$

We have shown that $L_X Y(q) = [X,Y](q)$ for all $q \in B$. By continuity it also hold in $\overline{B}$. It remains to show the proposition for the case $X = 0$ in a neighbourhood of p. In this case the equality $L_X Y(p) = 0 = [X,Y](p)$ is trivial. □

Another geometric interpretation of the Lie bracket is that it measures the noncommutativity of flows:

Proposition 1.35. *For vector fields $X, Y \in \Gamma(TM)$ the following assertions are equivalent:*

1. $[X,Y] = 0$.
2. ϕ^X *and* ϕ^Y *commute for small times in the following sense: For $p \in M$ there exists an open neighbourhood U of p and $\varepsilon > 0$ such that $\phi_t^X \phi_s^Y = \phi_s^Y \phi_t^X$ in U for all $s, t \in [-\varepsilon, \varepsilon]$.*

Remark. If X, Y are complete vector fields and the flows commute for small times, then they commute $\phi_s^X \phi_t^Y = \phi_t^Y \phi_s^X$ for all $s, t \in \mathbb{R}$.

Proof of proposition. So show '⇒', let $q \in U \subset M$ and $t_0 \in (-\varepsilon, \varepsilon)$. Then we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0} \mathrm{d}\phi_{-t}^X(Y \circ \phi_t^X(q)) &= \mathrm{d}\phi_{-t_0}^X \frac{\mathrm{d}}{\mathrm{d}h}\bigg|_{h=0} \mathrm{d}\phi_{-h}^X(Y \circ \phi_h^X(\phi_{t_0}^X(q))) \\ &= \mathrm{d}\phi_{-t_0}^X L_X Y(\phi_{t_0}^X(q)) = 0\end{aligned}$$

Hence $\mathrm{d}\phi_{-t}^X(Y \circ \phi_t^X(q)) = Y(q)$ for all $t \in (-\varepsilon, \varepsilon)$. We claim that ϕ_t^X sends trajectories of Y in trajectories of Y. This holds since $\phi_t^X \circ c_q^Y(0) = \phi_t^X(q)$ and

$$(\phi_t^X \circ c_q^Y)^\bullet(s) = \mathrm{d}\phi_t^X(\dot{c}_q^Y(s)) = \mathrm{d}\phi_t^X(Y(c_q^Y(s)) = Y(\phi_t^X(c_q^Y(s))) = Y(\phi_t^X \circ c_q^Y(s))$$

, so $\phi_t^X \circ c_q^Y = c_{\phi_t^X(q)}^Y$ is a trajectory of Y through $\phi_t^Y(q)$. That is

$$\phi_t^X \circ \phi_s^Y(q) = c_{\phi_t^X(q)}(s) = \phi_s^Y \circ \phi_t^X(q).$$

Conversely, if for (p,s,t) in a neighbourhood of $(q,0,0) \in M \times \mathbb{R} \times \mathbb{R}$ holds $\phi_s^Y \circ \phi_t^X(p) = \phi_t^X \circ \phi_s^Y(p)$, then $\phi_{-t}^X(c_{\phi_t^X(p)}(s)) = \phi_s^Y(p) = c_p^Y(s)$ and derivation with respect to s at $s = 0$ gives

$$\mathrm{d}\phi_{-t}^X(Y \circ \phi_t^X(p)) = Y(p)$$

which again derived wrt. t at $t = 0$ yields

$$[X, Y](p) = L_X Y(p) = 0. \qquad \square$$

1.6 Distributions and foliations

1.6.1 Foliations

An *immersed submanifold* of a manifold M is the image of an injective immersion $\iota\colon N \to M$ of a manifold N. It relaxes the notions of an (embedded) submanifold. Recall: $\iota(N)$ is an embedded submanifold if and only if ι is a homeomorphism into its image.

Definition 1.36. A k–dimensional *foliation* of a manifold M^m is a partition of M in disjoint k–dimensional submanifolds (called the *leaves* of the foliation). This partition is locally trivial in the following sense: At every point $x \in M$ there is a local chart of the form $\kappa\colon U \to D^k \times D^{m-k}$, where D^k is a k–dimensional disk, such that the intersection of each leaf with U is a (necessarily countable) union of 'layers' of the form $\kappa^{-1}(D^k \times \mathrm{pt})$.

In other words: The partition must be locally equivalent (via local diffeomorphism) to the model foliation of $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$ by the leaves $\mathbb{R}^k \times \mathrm{pt}$.

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One can regard foliations as geometric structures on manifolds givenby atlases with certain restrictions on the coordinate changes: A k–dimensional foliation on M (with connected leaves) corresponds to a maximal subatlas of the differentiable structure whose coordinate changes are of the form

$$\mathbb{R}^k \times \mathbb{R}^{m-k} \ni (x, y) \mapsto (f(x, y), g(y))$$

Starting from such an atlas, one recovers the (connected) leaves as equivalence classes of 'horizontally connectable' points: $x \sim y$ if and only if there exists a curve c from x to y such that wrt. the foliation charts κ the $\mathbb{R}^{m-k}$–component of $\kappa \circ c$ is locally constant.

Equivalently, one can recover the leaves as path components of the finer topology on M generated by the subsets $\kappa^{-1}(U \times \mathrm{pt})$ with $U \subset D^k$ open. ('transverse discretization').

Example.

0. Products $M = M_1 \times M_2$ is a foliation by the leaves $M_1 \times \mathrm{pt}$.
1. Submersions $f\colon M^m \to N^n$. The level sets $f^{-1}(\mathrm{pt})$ form an $(m-n)$–dimensional foliation of M as a consequence of the implicit function theorem. The leaves are embedded submanifolds.
2. Let X be a vector field on M without zeros. The (traces of the) trajectories form a 1–dimensional foliation. The leaves are in general not embedded.

Definition 1.37. A foliation is called a *fiber bundle* if

1. All leaves (also called *fibers*) are diffeomorphic to a fixed manifold ('model fiber') F,
2. The foliation is *locally trivial* in transverse direction: Every leaf has a saturated (i.e. a union of leaves) open neighbourhood U on which the foliation is a product foliation, i.e. there exist bundle charts $\kappa U^m \to F^k \times S^{m-k}$ such that the subsets $\kappa^{-1}(F \times \mathrm{pt})$ are leaves.

The *space of leaves* B carries a natural strcuture as a smooth manifold such that the natural projection $\pi\colon M \to B$ becomes a submersion. (In particular, B carries the quotient topology wrt. π). Namely, the bundle charts κ induce charts $\overline{\kappa}\colon \pi(U) \to S$ and the coordinate changes $\overline{\kappa}' \circ \overline{\kappa}^{-1}$ are defined on open subsets and are smooth. B is clearly Hausdorff and therefore a smooth manifold.

We call F the *fiber*, M the *total space* and B the *base space* and write the fiber bundle like this:

$$\begin{array}{ccc} F & \longrightarrow & M \\ & & \big\downarrow \pi \\ & & B \end{array}$$

Example.

0. The product foliation $\pi\colon F \times B \to B$ is a fiber bundle.
1. The tangent bundle $\pi\colon TM \to M$ of an m–dimensional manifold M is a bundle with fiber $\mathbb{R}^m$. Here, the fiber also carries an algebraic structure as an m–dimensional $\mathbb{R}$–vector spaces, varying smoothly. So the tangent bundle is a *vector bundle.*
2. The *Hopf fibration* is the fiber bundle

$$\begin{array}{ccc} \mathbb{C} \supset S^1 & \longrightarrow & S^{2n-1} \subset \mathbb{C}^n \\ & & \big\downarrow \pi \\ & & \mathbb{C}\mathrm{P}^{n-1} \end{array}$$

The fibers are the trajectories of the vector field $z \mapsto iz$. We have the following bundle charts:

$$U_i = \{z \in S^{2n-1} \mid z_i \neq 0\} \to S^1 \times \mathbb{C}^{n-1} \qquad z \mapsto \left(\frac{z_i}{|z_i|}, \frac{z_1}{z_i}, \ldots, \widehat{\frac{z_i}{z_i}}, \ldots, \frac{z_n}{z_i}\right)$$

1.6.2 Distributions

Definition 1.38. A k–dimensional *distribution* on a manifold M is a family $\mathcal{D} = (D_p)_{p \in M}$ of k–dimensional vector subspaces $D_p \subset T_pM$ which depend smoothly on p in the sense: For every $p \in M$ exist smooth vector field $X_1, \ldots, X_k$ near p such that

$$D_x = \mathrm{span}\{X_1(x), \ldots, X_k(x)\}$$

In later terminology, distributions are smooth *vector subbundles* of the tangent bundle TM. 1–dimensional distributions are called *line fields* and are locally generated by nonvanishing vector fields, $D_p = \mathbb{R}X(p)$ locally.

Example.

1. Distributions tangent to foliations
2. Let us fix a scalar product on $\mathbb{R}^m$. A distribution $\mathcal{D} = (D_p)_{p\in U}$ on an open subset $U \subset \mathbb{R}^m$ determines the orthogonal complementary distributions $\mathcal{D}^\perp = (D_p^\perp)_{p\in U}$ on U, e.g. if X is a nonvanishing vector field, then $\operatorname{span}\{X(p)\}^\perp$ is a hyperplane field or if f is a function without critical points, then $\mathbb{R}(\operatorname{grad} f)$ is a line field. The *standard contact structure* on S^{2n-1} is $z \mapsto (\mathbb{C}z)^\perp$, which is orthogonal to the Hopf fibration.

Definition 1.39. An immersed submanifold, more generally an immersion $\iota\colon N \looparrowright M$ is *tangential* to the distribution $\mathcal{D}$, if

$$d\iota(T_xN) \subset D_{\iota(x)} \subset T_{\iota(x)}M$$

for all $x \in N$. It is called an *integral manifold* if it has maximal dimension k, i.e. $d_\iota(T_xN) = D_{\iota(x)}$.

Definition 1.40. A distribution $\mathcal{D}$ is called *(completely) integrable*, if there exist integral manifolds through all points. It is called partially integrable if there are everywhere tangential submanifold of dimension ≥ 2.

Remark. Line fields are always integrable.

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Definition 1.41. The distribution $\mathcal{D} = (\mathcal{D}_p)_{p\in M}$ is (completely) integrable, if there exist (locally) integrable submanifolds through all points.

we observe: If X is a (locally) defined vector field tangent to the distribution $\mathcal{D}$, then the flow Φ of X preserves integral submanifolds N of the distribution (locally for small times), since X is tangent to N and Φ restricts to a flow on N. Infinitesimally, this means: For any two vector fields X and Y tangent to $\mathcal{D}$, the Lie derivative $L_XY = [X, Y]$ must again be tangent to any integral submanifold, because it can be computed intrinsically, i.e. along any integral submanifold of $\mathcal{D}$. This yields an obstruction to integrability of distributions of dim ≥ 2:

Definition 1.42. $\mathcal{D}$ is called involutive, if X,Y tangent to $\mathcal{D}$ $\Rightarrow [X, Y]$ tangent to $\mathcal{D}$.

Hence: integrable $\Rightarrow$ involutive.
distributions are in general not integrable:
Example: "propeller": We regard a time dependent distribution $\mathcal{D}$ on M as a distribution on $M \times \mathbb{R}$:

$$\hat{\mathcal{D}}(p, t) := \mathcal{D}_p(t) \oplus \mathbb{R} \subset T_pM \oplus T_t\mathbb{R} \cong T_{(p,t)}(M \times \mathbb{R})$$

Then $\hat{\mathcal{D}}$ is integrable and involutive if and only if $\mathcal{D}(t)$ is stationary (= time independent)

Theorem 1.43. *(Frobenius:) involutive $\Rightarrow$ tangent to a foliation ($\Rightarrow$ (completely) integrable [leaves are integrable submanifolds])*

Proof. Let $X_1, \ldots, X_k$ be a local basis for $\mathcal{D}$ (i.e. locally defined vector fields s.t. $\{X_i(p)\}$) is a casis of $\mathcal{D}_p$.
Involutivity implies that along X_1-trajectories we habe an ODE

$$L_{X_1} X_j = \sum_{i=1}^{k} a_{ij} X_i,$$

where a_{ij} are components of a $k \times k$ matrix A that depends smoothly on the point. In matrix notation, this gives

$$L_{X_1} \underbrace{X}_{(X_1, \ldots X_k)} = XA \quad (*)$$

We choose a (piece of) hypersurface (= codim-1 submf.) S transversal to X_1 and solve locally the auxiliary ODE:

$$\partial_{X_1} B + AB = 0$$

for $k \times k$-matrix valued functions B along X_1-trajectories with initial condition $B|_S = \mathrm{id}$.
Then:

$$L_{X_1}(\underbrace{XB}_{\sum_i X_i b_{i1}, \cdots \sum X_i b_{ik}}) = (\underbrace{L_{X_1}, X}_{XA})B + X(\underbrace{\partial_{X_1} B}_{-AB}) = 0$$

This means that the flow of X_1 preserves the distribution $\mathcal{D}$.
(Variant with $\wedge$-product:

$$\begin{aligned} L_{X_1}(X_1 \wedge \cdots \wedge X_L) &= \underbrace{(L_{X_1} X_1)}_{\sum_i a_{i1} X_i} \wedge X_2 \wedge \cdots \wedge X_k + \cdots + X_1 \wedge X_2 \wedge \cdots \wedge \underbrace{L_{X_1} X_k}_{\sum_i a_{ik} X_i} \\ &= \mathrm{tr} A \cdot X_1 \wedge X_2 \wedge \cdots \wedge X_k \\ &\Rightarrow \Lambda_k TM \supset \mathbb{R} \cdot X_1 \wedge \cdots \wedge X_t \text{ stationary along } X_1\text{-flow}) \end{aligned}$$

In suitable local coordinates we have $X_1 = \frac{\partial}{\partial x_1}$ and $S = \{X_1 = 0\}$, and $\mathcal{D}$ does not depent on X_1 ("propeller stands still").
This implies: $\mathcal{D}$ is locally integrable if and only if the induced involutive distribution $(\mathcal{D}_p \cap T_p S)_{p \in S}$ on S is locally integrable. This reduces the problem by one dimension (i.e. $\mathrm{Frob}_k \Leftarrow \mathrm{Frob}_{k-1}$). The assertion follows by induction on k from the trivial one-dim case (note: also the 1-dim involutivity condition is trivial, $L_X X = [X, X] = 0$). □

"classical" coordinate version of Frobenius: $U \subset \mathbb{R}^m, V \subset \mathbb{R}^n$ open. Local solutions: $U \supset U_0 \xrightarrow{\alpha} V$ of the total differential equation

$$d\alpha_x = b(x, \alpha(x))$$

for given $U \times V \xrightarrow{b} \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ correspond to integral submanifolds of the distribution $(x, y) \to$ graph $b(x, y)$ on $U \times V$. The distribution is spanned by the vector fields

$\frac{\partial}{\partial x_j} + \sum_i b_{ij}\frac{\partial}{\partial y_i}$, and its involutivity corresponds to the integrability condition for b:

$$\left(\frac{\partial b_{kl}}{\partial x_j} - \frac{\partial b_{kj}}{\partial x_l}\right) + \sum_i \left(b_{ij}\frac{\partial b_{kl}}{\partial y_i} - b_{il}\frac{\partial b_{kj}}{\partial y_i}\right) = 0 \quad \forall k,j,l \quad \text{(Exercise!)}$$

Special case: One can locally find a ($\mathbb{R}$-valued) function d with prescribed differential

$$d\alpha = \beta \ (= \sum \beta_i dX_i)$$

if and only if the integrability holds:

$$\frac{\partial \beta_l}{\partial x_j} - \frac{\partial \beta_j}{\partial x_l} = 0 \quad \forall j,l$$

The computation

$$L_{X_i}(f_j X_j) = \underbrace{(X_i f_j)X_j}_{\text{cont. in } \mathcal{D}} + f_j L_{X_i} X_j$$

Shows for vector fields X and Y tangent to $\mathcal{D}$: $(L_X Y)(p)$ mod $\mathcal{D}_p$ depends only on $X(p)$ and $Y(p)$. As obstruction to involutivity and hence integrability of $\mathcal{D}$, we therefore obtain the family of skew-symmetric bilinear maps:

$$\begin{aligned} \mathcal{D}_p \times \mathcal{D}_p &\longrightarrow T_pM/_{\mathcal{D}_p} \\ (X(p), Y(p)) &\longrightarrow [X,Y](p) \text{ mod } \mathcal{D}_p \end{aligned}$$

(tensor: vector valued 2 form on the distribution $\mathcal{D}$).

04.12.2012

1.7 The cotangent bundle and 1–forms

1.7.1 The cotangent bundle

By fiberwise linear algebra constructions new vector bundles arise from given ones. The construction of the dual vectorspace starting from the tangent bundle $\pi TM \to M$ yields the *cotangent bundle.*

$$T^*M = \coprod_{p \in M} T_p^*M \xrightarrow{\pi} M$$

The total differential of a function $f \in \mathscr{C}^1(M)$ in a point $p \in M$ is a cotangent vector: $\mathrm{d}f_p$ defined by $\mathrm{d}f_p(v) = \partial_v f$ for any $v \in T_pM$. The differential $\mathrm{d}f_p$ is an element of the cotangent space T_p^*M.

The natural structure on T^*M as a smooth manifold is (as in the case of the vector bundle TM) induced by the smooth structure on M. If (U,x) is a chart for M around p, then the differentials $(\mathrm{d}x_i)_p$ of the coordinate functions form a basis of T_p^*M which is dual to the basis $\{\partial_i|_p\}$ of T_pM, because

$$\mathrm{d}f_p(\partial_i) = \frac{\partial(f \circ x^{-1})}{\partial x_j}(x(p)) \quad \Rightarrow \quad (\mathrm{d}x_i)_p(\partial_i) = \frac{\partial x_i}{\partial x_j}(x(p)) = \delta_{ij}$$

The induced chart for T^*M is

$$T^*x\colon T^*U = \coprod_{p\in U} T_p^*M \longrightarrow x(U)\times\mathbb{R}^{\dim M}\subset\mathbb{R}^{2\dim M},\quad a_i(\mathrm{d}x_i)_p\mapsto (x(p),\textstyle\sum_i a_i e_i)$$

The change of basis wrt. a coordinate change is given by

$$\mathrm{d}x_i = \sum_j \left(\frac{\partial x_i}{\partial \widetilde{x_j}}\circ\widetilde{x}\right)\mathrm{d}\widetilde{x_j}$$

The induced coordinate change on T^*M is then given by

$$(x,a)\mapsto\left(\widetilde{x}(X),\sum_{i,j}\left(a_i\frac{\partial x_i}{\partial\widetilde{x_j}}(\widetilde{x}(x))\right)e_j\right)$$

As before in the case of TM, we conclude that T^*M is a $\mathscr{C}^\infty$ manifold and π is smooth.

1.7.2 1–forms

Definition 1.44. A smooth *differential form of degree 1* (also 1–form or Pfaffian form) on M is a smooth section of $\pi\colon T^*M\to M$, i.e. a smooth map $\alpha\colon M\to T^*M$ with $\pi\circ\alpha=\mathrm{id}_M$, i.e. $\alpha_p\in T_p^*M$.

Example. Let $f\in\mathscr{C}^\infty(M)$. The total differential $\mathrm{d}f\colon p\to\mathrm{d}f_p$ is a smooth 1–form.

In local coordinates, we write 1–forms as

$$\alpha_U=\sum_i a_i\,\mathrm{d}x_i$$

The smoothness of a $\alpha|_U$ amounts to the smoothness of the coefficients a_i. For instance,

$$\mathrm{d}f|_U=\sum_i\left(\frac{\partial(f\circ x^{-1})}{\partial x_i}\circ x\right)\mathrm{d}x_i$$

The duality of tangent and cotangent vecotrs corresponds to a duality of vector fields and 1–forms: The space $\Omega^1(M)$ of smooth 1–foms on M is a $\mathscr{C}^\infty(M)$–module:

$$(f\alpha)_p=f(p)\alpha_p,\quad f\in\mathscr{C}^\infty(M),\alpha\in\Omega^1(M)$$

There is the natural $\mathscr{C}^\infty(M)$–bilinear pairing

$$\Omega^1(M)\times\Gamma(TM)\to\mathscr{C}^\infty(M),\quad(\alpha,X)\mapsto\alpha(X)$$

where $\alpha(X)_p\coloneqq\alpha_p(X_p)$. The smoothness of $\alpha(X)$ becomes clear in local coordinates:

$$\left(\sum_i a_i\,\mathrm{d}x_i\right)\left(\sum_j b_j\frac{\partial}{\partial x_j}\right)=\sum_i a_ib_i$$

This pairing is non–degenerate, i.e. the canonical map

$$\Omega^1(M) \to \mathrm{Hom}_{\mathscr{C}^\infty(M)}(\Gamma(TM), \mathscr{C}^\infty(M)), \quad \alpha \mapsto (X \mapsto \alpha(X))$$

is bijective. Hence we can interpret 1–forms as $\mathscr{C}^\infty(M)$–linear operators on vector fields and vice versa.

One can pull back 1–forms with $\mathscr{C}^\infty$–maps maps $F\colon M \to N$:

$$\mathrm{d}F_p\colon T_pM \to T_{F(p)}N \quad \rightsquigarrow \quad \mathrm{d}F_p^*\colon T^*_{F(p)}N \to T^*_pM, \quad \lambda \mapsto \lambda \circ \mathrm{d}F_p$$

Pointwise application of this process to a 1–form yields:

$$F^*\colon \Omega^1(N) \to \Omega^1(M), \quad \alpha \mapsto (p \mapsto \alpha \circ \mathrm{d}F_p)$$

The map F^* is called the *pull–back*. For vector fields $X \in \Gamma(TM)$ the following holds:

$$(F^*\alpha)(X) = \alpha(\,\mathrm{d}F(X))$$

The right hand side is always smooth, and we see that $F^*\alpha$ is a smooth section of T^*M.

Remark. There is no well–defined push–forward for vector fields.

1.7.3 The line integral

Let $\omega = g(t)\,\mathrm{d}t$ be a piecewise continuous 1–form on an interval $[a, b]$, i.e. g is a piecewise smooth function on $[a, b]$. One can integrate ω over the (oriented) interval by defining

$$\int_{[a,b]} \omega := \int_a^b g(t)\,\mathrm{d}t$$

Lemma 1.45 (Diffeomorphism invariance)**.** *For an orientation preserving diffeomorphism $\varphi\colon [\tilde{a}, \tilde{b}] \to [a, b]$ holds*

$$\int_{[\tilde{a},\tilde{b}]} \varphi^*\omega = \int_{[a,b]} \omega$$

and the sign changes for an orientation reversing diffeomorphism.

Proof.

$$\begin{aligned}\int_{[\tilde{a},\tilde{b}]} \varphi^*\omega &= \int_{\tilde{a}}^{\tilde{b}} (\varphi^*\omega)\left(\frac{\partial}{\partial \tilde{t}}\Big|_{\tilde{t}}\right)\,\mathrm{d}\tilde{t} = \int_{\tilde{a}}^{\tilde{b}} \omega\left(\mathrm{d}\varphi\frac{\partial}{\partial \tilde{t}}\Big|_{\tilde{t}}\right)\,\mathrm{d}\tilde{t} \\ &= \int_{\tilde{a}}^{\tilde{b}} g(\varphi(\tilde{t}))\varphi'(\tilde{t})\,\mathrm{d}\tilde{t} = \pm\int_a^b g(t)\,\mathrm{d}t = \pm\int_{[a,b]} \omega\end{aligned}$$

□

The *line integral* of a (continuous) 1–form α on M over a (piecewise $\mathscr{C}^1$) curve $c\colon [a, b] \to M$ is defined as

$$\int_c \alpha := \int_{[a,b]} c^*\alpha = \int_a^b \alpha(\dot{c}(t))\,\mathrm{d}t$$

As a consequence of the lemma, the line integral is invariant under reparametrisation:

$$\int_{c\circ\varphi}\alpha = \int_{[\tilde a,\tilde b]}(c\circ\varphi)^*\alpha = \pm\int_{[a,b]}c^*\alpha = \pm\int_c\alpha$$

Thus, we can integrate 1–forms over unparametrized oriented curves, e.g. oriented 1–submanifolds with no additional structure.

Example. Define the following 1–form on $\mathbb{R}^2 \smallsetminus \{0\}$

$$\alpha = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2+y^2}$$

and consider the curve $c(t) = (\cos t, \sin t)$ on $[0, 2\pi]$. Then

$$\dot c(t) = -\sin t\frac{\partial}{\partial x}\Big|_{c(t)} + \cos t\frac{\partial}{\partial y}\Big|_{c(t)}$$

and thus $\alpha(\dot c(t)) = \sin^2 t + \cos^2 t = 1$, so

$$\int_c\alpha = \int_0^{2\pi}\mathrm{d}t = 2\pi$$

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Definition 1.46. A 1-form α is called exact if it is the total differential of a function $f:\ \alpha = df$.

The "fundamental theorem of calculus" yields for the line integral of exact forms: "fundamental thm. for the line integrals:"
For $f \in \mathcal{C}^1(M)$ and $[a,b] \xrightarrow[\text{p.w. }\mathcal{C}^1]{c} M$ holds:

$$\int_c df = f\big|_{c(a)}^{c(b)}\ (:= f(c(b)) - f(c(a)))$$

because:

$$\int_c df = \int_a^b \underbrace{c^*df}_{d(f\circ c)=(f\circ c)'(t)dt} = \int_a^b (f\circ c)'(t)dt = (f\circ c)\big|_a^b$$

In this case, the line integral is independent of the path (curve) (rel endpoints) i.e. its value depends only on the position of the endpoints.

Proposition 1.47. *The line integral of $\alpha \in \Omega'(M)$ is path independent (rel. endpoints) if and only if α is exact.*

Proof. " $\Rightarrow$ " w.l.o.g. let M be connected. Then any two points are connected by a piecewise $\mathcal{C}^1$-curve. We choose a basepoint p and obtain a well-defined function f ("potential") by putting

$$f(x) := \int_{c_{p,x}}\alpha,$$

where $c_{p,x}$ is an arbitrary piecewise $\mathcal{C}^1$-curve from p to x. To see that f is $\mathcal{C}^1$ with $df = \alpha$, we compute (in loc coordinates:)

$$f(x+v) = f(x) + \int_0^1 \alpha_{x+tv}(v)dt = f(x) + \alpha_x(v) + \int_0^1 \underbrace{(\alpha_{x+tv} - \alpha_x)(v)}_{=o(\|v\|);\ \to 0 \text{ uniformly as } v\to 0} dt$$

$\Rightarrow df_x = \alpha_x$ □

Remark on physics (mechanics):
$\mathbb{R}^3 \supset U$ (open) $\xrightarrow{F} \mathbb{R}^3$ force field $\rightsquigarrow \alpha = \langle F, \bullet \rangle$ (standard SCP on $\mathbb{R}^3$); $[a,b] \xrightarrow{c} U$ path.
$\rightsquigarrow$ work which the force field does along the path equals

$$W = \int_a^b \langle F, \dot{c} \rangle dt = \int_a^b \alpha(\dot{c})dt = \int_c \alpha$$

F conservative $\Leftrightarrow$ no work along closed paths: "conservation of mechanical energy" for all closed curves c:

$$\int_c \alpha = 0$$

$\Leftrightarrow \exists$ potential $U : \alpha = -dU$, resp. $F = -\text{grad } U$
work along arbitrary c : $\int_c \alpha = U(c(a)) - U(c(b))$
infinitesimally: $\dot{w} = \alpha(\dot{c}) = -\dot{U}$ (decrease of potential)

Example: gravitational field, electric field: $\alpha = -\frac{dr}{r^2} = d\frac{1}{r} \ \Rightarrow U = -\frac{1}{r}$
If c satisfies the equation of motion $\langle \ddot{c}, \bullet \rangle = \alpha$

$$\Rightarrow -\dot{E}_{pot} = \dot{W} = \alpha(\dot{c}) = \langle \ddot{c}, \dot{c} \rangle = \frac{1}{2}\langle \dot{c}, \dot{c} \rangle^{\cdot} = \dot{E}_{kin}$$
$$\dot{E}_{pot} + \dot{E}_{kin} = 0 \qquad \text{conservation of pot + kin energy}$$

Criteria of exactness of a 1-form $\alpha \in \Omega'(M)$, i.e. for the solvability of the PDE

$$df = \alpha \qquad (*)$$

in local coordinates:

$$\frac{\partial f}{\partial x_i} = \alpha_i.$$

Solutions of $(*)$ can be interpreted as integral manifolds of a codim-1 distribution on $M \times \mathbb{R}$ (compare the earlier discussion of the "classical" Frobenius thm.), namely $D(p,t) =$ graph α_p module the identification $T_(p,t)(M \times \mathbb{R}) \cong T_pM \oplus \mathbb{R}$. The local solvability of $(*)$ is according to the Frobenius Thm. equivalent to the involutivity of $\mathcal{D} = (D_{(p,t)})$, which corresponds to an integrability condition for α.

We word out the obstruction for involutivity in this case:
Vector fields X on M yield vector fields tangent to $\mathcal{D}$ by

$$(p,t) \longrightarrow X(p) + \alpha(X(p))\frac{\partial}{\partial t}|_{(p,t)}$$

$$\text{short:} \qquad \tilde{X} + \alpha(X)E \qquad \text{where } \tilde{X} \text{ is the horizontal lift.}$$

We compute the Lie brackets of these vector fields mod $\mathcal{D}$.

$$[\tilde{X} + \alpha(X)E, \tilde{Y} + \alpha(Y)E] = [\tilde{X}, \tilde{Y}] + (X(\alpha(Y)) - Y(\alpha(X)))E,$$

because E commutes with $\tilde{X}$ and $\tilde{Y}$. With $[\tilde{X}, \tilde{Y}] = [X,\tilde{Y}] = -\alpha([X,Y])E$ mod $\mathcal{D}$ it follows:

$$\text{l.h.s } = \underbrace{(X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X,Y]))}_{:=d\alpha(X,Y)} E \text{ mod } \mathcal{D}$$

As remarked in our discussion of Frobenius, the obstruction $[U,V]$mod $\mathcal{D}$ for vector fields U and V tangent to $\mathcal{D}$ is of 0. order in U and V, i.e. its value is a skew-symmetric bilinear form in $U(p)$ and $V(p)$. Hence the well-defined smooth family $d\alpha = ((d\alpha)_p)_{p\in M}$ of skew-symmetric bilinear forms $d\alpha_p$ on T_pM given by

$$(d\alpha)_p(X(p), Y(p)) = d\alpha(X,Y)(p)$$

is the complete obstruction to local exactness of α. (this is one motivation for introducing higher degree differential forms and the exterior derivative $\rightarrow$ later)

Definition 1.48. $\alpha \in \Omega'(M)$ is called <u>closed</u>, if $d\alpha = 0$.

Thus: locally exact $\Leftrightarrow$ closed (for 1-forms)
In local coordinates:

$$\alpha = \sum \alpha_i \, dx_i \Rightarrow d\alpha\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j}$$

Here, the local integrability conditions for $\frac{\partial f}{\partial x_i} = \alpha_i$ are

$$\frac{\partial \alpha_j}{\partial x_i} - \frac{\partial \alpha_i}{\partial x_j} = 0 \qquad \forall i,j$$

Integrability condition for a force field F to be locally conservative:

$$\frac{\partial F_j}{\partial x_i} - \frac{\partial F_i}{\partial x_j} = 0$$

Remark:
One can represent $d\alpha$ as a measure for local nonexactness of α geometrically also as follows:

$$(d\alpha)_p(u,v) = \lim_{t\to 0} \frac{1}{t^2} \int_{\gamma_t} \alpha \qquad \text{Exercise!}$$

Te obstructions to the global solvability of (*) for closed 1-forms α are of topologlical nature. This results from
Homotopy invariance of the line integral of closed 1-forms:
$[0,1]\times[a,b] \xrightarrow{\mathcal{C}^1} M : \ (s,t) \to c_s(t)$ homotopy with fixed end points $c_0(a) :=$ const, $c_0(b) :=$ const

$$d\alpha = 0 \Rightarrow s \to \int\limits_{c_s} \alpha \text{ const} \qquad \text{proof: Exercise (use local potential)}$$

Whether the line integral of closed forms is path independent is related to the possibility of deforming (homotoping) paths into each other (fixing endpoints). In case of simple topology: closed $\Rightarrow$ exact (no global obstruction).

11.12.2012

Whether the line integral of closed 1–forms is path independent is related to the possibility of deforming paths in M into each other. This is related to the *fundamental group* $\pi_1(M)$. Precisely, if all closed paths in M are nullhomotopic (i.e. homotopic relative to their endpoints to the constant path), then the line integral of closed 1–forms is path independent, and closed 1–forms are therefore exact. Short: If $\pi_1(M) = 0$, then $H^1(M) = 0$, where H^1 is the first de–Rham cohomology group.

Thus, if M has sufficiently simple topology, then there are no global obstructions to exactness.

Example (Poincaré–Lemma for 1–forms)**.** For $U \subset \mathbb{R}^n$ open star–shaped the following holds: If $\alpha \in \Omega^1(U)$ is closed, then it is exact.

Example. Consider polar coordinates on $\mathbb{R}^2 \setminus \{0\}$. The 1–form $\mathrm{d}\theta \subset \Omega^1(\mathbb{R}^2 \setminus \{0\})$ is well–defined and closed, hence locally exact. But it is not globally exact since for the circle γ around the origin, we have

$$\int_\gamma \mathrm{d}\theta = 2\pi \neq 0$$

1.8 Digression into multilinear algebra

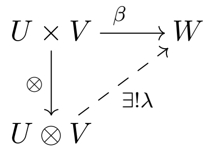
1.8.1 Tensor product of vector spaces

We work with vector spaces and algebras over a fixed but arbitrary field K (while thinking of K as being $\mathbb{R}$ or $\mathbb{C}$). We define the tensor product of vector spaces by a universal property:

Definition 1.49. A *tensor product* of two vector spaces U and V is a vector space $U \otimes V$ together with a bilinear map

$$\otimes\colon U \times V \to U \otimes V, \quad (u,v) \mapsto u \otimes v$$

which has the following universal property:



For any bilinear map $\beta\colon U \times V \to W$ there exists a unique linear map $\lambda\colon U \otimes V \to W$ with $\beta = \lambda \circ \otimes$.

In other words, the natural homomorphism of vector spaces

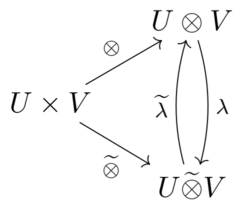
$$\mathrm{Hom}(U \otimes V, W) \to \mathrm{Bil}(U, V; W), \quad \lambda \to \lambda \circ \otimes$$

is an isomorphism.

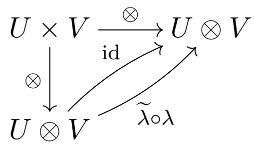
Theorem 1.50. *For any vector spaces U, V the tensor product $\otimes\colon U \times V \to U \otimes V$ exists and is unique up to natural isomorphism. We can therefore speak of* the *tensor product of U and V.*

Proof. The uniqueness follows from the universal property: It yields linear maps λ and $\widetilde{\lambda}$ with

$$\lambda(u \otimes v) = u \,\widetilde{\otimes}\, v \qquad \widetilde{\lambda}(u \,\widetilde{\otimes}\, v) = u \otimes v \quad \Rightarrow \quad (\widetilde{\lambda} \circ \lambda)(u \otimes v) = u \otimes v$$



Both $\mathrm{id}_{U \otimes V}$ and $\widetilde{\lambda} \circ \lambda$ solve the mapping problem



and uniqueness of its solution implies $\widetilde{\lambda} \circ \lambda = \mathrm{id}_{U \otimes V}$. Analogously, $\lambda \circ \widetilde{\lambda} = \mathrm{id}_{U \widetilde{\otimes} V}$. This shows that there are *natural* isomorphisms between and two tensor products of U and V.

For existence, we first consider a vector space E with basis $U \times V$, i.e. E consists of all finite linera combinations $\sum_i a_i(u_i, v_i)$ with $u_i \in U$, $v_i \in V$ and $a_i \in K$. In order to make the canonical map $U \times V \to E$ bilinear, we devide out corresponding relations: Let $R \subset E$ be the vector subspace spanned by the elements

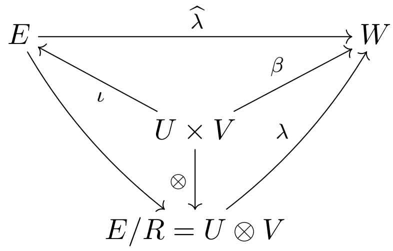
$$\begin{aligned}
(u_1, v) + (u_2, v) - (u_1 + u_2, v) &\qquad \forall u_1, u_2 \in U,\ v \in V \\
(u, v_1) + (u, v_2) - (u, v_1 + v_2) &\qquad \forall u \in U,\ v_1, v_2 \in V \\
(\alpha u, v) - \alpha(u, v) &\qquad \forall u \in U,\ v \in V,\ \alpha \in K \\
(u, \alpha v) - \alpha(u, v) &\qquad \forall u \in U,\ v \in V,\ \alpha \in K
\end{aligned}$$

Then the composition

$$\underbrace{U \times V \longrightarrow E \twoheadrightarrow}_{\otimes} E/R = U \otimes V$$

because, for instance $u_1 \otimes v + u_2 \otimes v - (u_1 + u_2) \otimes v = 0$. It satisfies the required universal property: Given a bilinear map $\beta\colon U \times V \to W$, there exists a unique linear map $\widehat{\lambda}\colon E \to W$ with $\widehat{\lambda} \circ \iota = \beta$. The bilinearity of β is equivalent to $\widehat{\lambda}(R) = 0$, because e.g.

$$\widehat{\lambda}((\alpha u, v) - \alpha(u, v)) = \widehat{\lambda}((\alpha u, v)) - \alpha\widehat{\lambda}((u, v)) = \beta(\alpha u, v) - \alpha\beta(u, v) = 0$$



Therefore $\widehat{\lambda}$ descends to a linear map $\lambda\colon U \otimes V \to W$ which satisfies as desired:

$$\lambda(u \otimes v) = \widehat{\lambda}((u, v)) = \beta(u, v) \quad \Rightarrow \quad \beta = \lambda \circ \otimes$$

This determines the map λ uniquely, because by our construction the $u \otimes v$ generate the vector space $U \otimes V$. □

The following result gives a more concrete idea of the tensor product.

Lemma 1.51. *If $\{e_i \mid i \in I\}$ is a basis of U and $\{f_i \mid i \in J\}$ is a basis of V, then $\{e_i \otimes f_j \mid (i, j) \in I \times J\}$ is a basis of $U \otimes V$. In particular, in the case of finite dimensions, we get*

$$\dim U \otimes V = \dim U \cdot \dim V$$

Proof. From the construction of the tensor product in the preceding proof we know that the elements $u \otimes v$ generate $U \otimes V$. They are linear combinations of the $e_i \otimes f_j$ because of the bilinearity of the tensor product:

$$\Big(\sum_i a_i e_i\Big) \otimes \Big(\sum_i b_i f_i\Big) = \sum_{i,j} a_i b_j \, e_i \otimes f_j$$

To verify their linear independence, we construct (using the universal property) linear forms 'seperating' them, i.e. taking independent values on them. The bilinear form (where e^* and f^* denote the dual basis elements, i.e. $e_k^*(e_i) = \delta_{ik}$, $f_l^*(f_j) = \delta_{jl}$)

$$U \times V \to K, \quad (u, v) \mapsto e_k^*(u) \cdot f_l^*(v)$$

induces the linear form

$$U \otimes V \to K, \quad u \otimes v \mapsto e_k^*(u) \cdot f_l^*(v) \quad e_i \otimes f_j \mapsto \delta_{ik}\delta_{jl}.$$

For a linear relation

$$\sum_{i,j} e_{ij} \, e_i \otimes f_j = 0$$

where $e_{ij} = 0$ for almost all (i, j) follows by applying the above linear form, that

$$0 = \sum_{i,j} c_{ij} = \delta_{ik}\delta_{jl} = c_{kl} \quad \forall k, l$$

Thus, the $e_i \otimes f_j$ form a basis. □

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Remark: To construct the tensor product of vector spaces, one can also proceed more directly and, after choosing bases $\{e_i\}$ of U and $\{f_j\}$ of V, construct $U \otimes V$ as the vector space consisting of the symbols $e_i \otimes f_j$ and define the natural bilinear map $U \times V \xrightarrow{\otimes} U \otimes V$ by

$$(\sum_i a_i e_i) \otimes (\sum_j b_j f_j) := \sum_{i,j} a_i b_j \; e_i \otimes f_j.$$

The universal property can easily be verified (Exercise).
The universal proberty implies the base independence for our second construction of the tensor product.
the abstract construction is more general. One can in the same way construct the tensor product $\otimes_R$ of mudules over a fixed ring R (e.g. $R = \mathbb{Z}$: abelian groups) also if the modules are not free, e.g. have torsion.
Change of bases:

$$\left.\begin{array}{l} \tilde{e}_r = \sum_i g_{ri} \; e_i \text{ in U} \\ \tilde{f}_s = \sum_j h_{sj} \; f_j \text{ in V} \end{array}\right\} \Rightarrow \tilde{e}_r \otimes \tilde{f}_s = \sum_{i,j} g_{rj} f_{sj} \; e_i \otimes f_j$$

$(\tilde{e}_r = g^i_r e_i, \; \tilde{f}_s = h^j_s f_j, \; \tilde{e}_r \otimes \tilde{f}_s = g^i_r h^j_s \; e_i \otimes f_j)$
Example: For vector spaces U and V , the bilinear map

$$\begin{aligned} U^* \times V &\longrightarrow \mathrm{Hom}(U, V) \\ (u^*, v) &\longrightarrow (u \longrightarrow u^*(u) \cdot v) \end{aligned}$$

induces a natural homomorphism (Ex: injective!)

$$U^* \otimes V \longrightarrow \mathrm{Hom}(U, V) \qquad (*)$$

(the image constists of homomorphisms $U \to V$ with finite-dim image)

$$\begin{array}{ccc} U^* \times V & \longrightarrow & \mathrm{Hom}(U, V) \\ \otimes \downarrow & \nearrow & \\ U^* \otimes V & & \end{array}$$

If $\dim U, \dim V < \infty$, then it is an isomorphism. Namely, if $\{e_i\}$ is a basis of U and $\{e_i^*\}$ the dual basis of U^*, then for arbitrary elements $v_i \in V$ the element $\sum_i e_i^* \otimes v_i \in U^* \otimes V$

corresponds to the homomophism $U \to V$ mapping $e_i \to v_i$.
Hence $(*)$ is surjective and therefore an isomorphism by dimesion reasons.
If $\{f_j\}$ is a basis for V, then

$$\sum_{j,i} a_{ji} f_j \; e_i^* \otimes f_j \qquad a_{ji} \in K$$

corresponds to the homomorphism $U \to V$ which is given relative to the chosen basis by the matrix $(a_{ji})_{j,i}$ because it maps $e_i \longrightarrow \sum_j a_{ji}\ f_j$ ($\longleftarrow$ i-th column).
In particular, the natural homomorphism $U^* \otimes U \longrightarrow \mathrm{End}(U)$ is an isomorphism if $\dim U < \infty$. (Note that then $\sum_i e_i^* \otimes e_i$ corresponds to id_U.)
Analogously to the twofold one defines and constructs the <u>multiple tensor product</u> $U_1 \otimes \cdots \otimes U_n$. It satisfies the universal property.

$$\begin{array}{ccc} U_1 \times ... \times U_n & \xrightarrow{\text{multilin.}} & W \\ {\scriptstyle \otimes}\big\downarrow{\scriptstyle \text{multilin.}} & \nearrow{\scriptstyle \exists!\ \text{lin (unique)}} & \\ U_1 \otimes ... \otimes U_n & & \end{array}$$

Lemma 1.52. *(Associativity:) For vector spaces* $U_1, \ldots, U_{n+m}$ $(n, m \geq 1)$ *there exists an unique isomophism of vector spaces with*

$$\begin{aligned} (U_1 \otimes \cdots \otimes U_n) \otimes (U_{n+1} \otimes \cdots \otimes U_{n+m}) &\longrightarrow U_1 \otimes \cdots \otimes U_{n+m} \\ (u_1 \otimes \cdots \otimes u_n) \otimes (u_{n+1} \otimes \cdots \otimes u_{n+m}) &\longrightarrow u_1 \otimes \cdots \otimes u_{n+m} \end{aligned}$$

Proof. The multilinear map

$$\begin{aligned} U_1 \times \cdots \times U_{n+m} &\longrightarrow (U_1 \otimes \cdots \otimes U_n) \otimes (U_{n+1} \otimes \cdots \otimes U_{n+m}) \\ u_1 \times \cdots \times u_{n+m} &\longrightarrow (u_1 \otimes \cdots \otimes u_n) \otimes (u_{n+1} \otimes \cdots \otimes u_{n+m}) \end{aligned}$$

induces the homomorphism

$$\begin{aligned} U_1 \otimes \cdots \otimes U_{n+m} &\longrightarrow (U_1 \otimes \cdots \otimes U_n) \otimes (U_{n+1} \otimes \cdots \otimes U_{n+m}) \\ u_1 \otimes \cdots \otimes u_{n+m} &\longrightarrow (u_1 \otimes \cdots \otimes u_n) \otimes (u_{n+1} \otimes \cdots \otimes u_{n+m}) \qquad (*) \end{aligned}$$

To see that it is an isomorphism, one can choose bases and observe that $(*)$ sends the induced basis bijectively to the induced basis. (Alternatively, one can use the universal property also to construct an inverse map). □

<u>Permutation of factors:</u> For vector spaces $U_1, \ldots U_n$ and a permutation $\sigma \in S_n$, there is the natural isomorphism of vector spaces:

$$\begin{aligned} U_1 \otimes \cdots \otimes U_n &\longrightarrow U_{\sigma(1)} \otimes \cdots \otimes U_{\sigma(n)} \\ u_1 \otimes \cdots \otimes u_n &\longrightarrow u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)} \end{aligned}$$

Functoriality: Homomorphisms of vector spaces $U_1 \xrightarrow{\alpha_1} V_1, \ldots, U_n \xrightarrow{\alpha_n} V_n$ induce the natural homomorphism

$$\begin{aligned} U_1 \otimes \cdots \otimes U_n &\xrightarrow{\alpha_1 \otimes \cdots \otimes \alpha_n} V_1 \otimes \cdots \otimes V_n \\ u_1 \otimes \cdots \otimes u_n &\longrightarrow \alpha_1(u_1) \otimes \cdots \otimes \alpha_n(u_n) \end{aligned}$$

Tensor algebra: We denote by $T_m(U) := \bigotimes^m U := \underbrace{U \otimes \cdots \otimes U}_{m}$ the m-fold tensor product of U with itself, $m \in \mathbb{N}_0$.
Convention: $T_0(U) := K$. Of course, $T_1(U) = U$. We consider the graded vector space

$$T_*(U) := \bigoplus_{m=0}^{\infty} T_m(U).$$

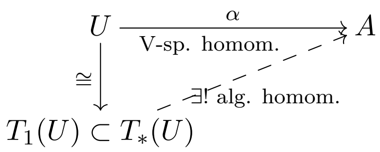
There are natural bilinear maps (by associativity)

$$\begin{aligned} T_m(U) \times T_n(U) &\xrightarrow[bil]{\otimes} T_{m+n}(U) \\ (u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_n) &\longrightarrow u_1 \otimes \cdots \otimes v_n \end{aligned}$$

i.e. $\otimes$ defines (after bilinear extension) a product

$$T_*(U) \times T_*(U) \xrightarrow[bil]{\otimes} T_*(U)$$

$T_*(U)$ becomes a graded associative K-algebra with unit element, the covariant tensor algebra. Covariant, because the functor $U \longrightarrow T_*(U)$ from vector spaces to algebras is covariant, i.e. a homomorphism of vector spaces $U \to V$ induces an algebra homomorphism $T_*(U) \longrightarrow T_*(V)$ in the same direction. Also the tensor algebra can be characterized by a universal property:
$T_*(U)$ is the "largest associative K-algebra with 1 generated by U", i.e. for every homomorphism of vector spaces $U \xrightarrow{\alpha} A$ into an associative K-algebra with 1 exists an unique extension to an algebra homomorphism $T_*(U) \to A$. It satisfies $u_1 \otimes \cdots \otimes u_n \longrightarrow \alpha(u_1) \cdot \ldots \cdot \alpha(u_n)$.



The covariant tensor algebra of U is defined as

$$T^*(U) := T_*(U^*) \qquad \text{(monomials of degree n } u_1^* \otimes \cdots \otimes u_n^+ \in T^n(U))$$

A homomorphism of vector spaces $U \to V$ induces a homomorphism of vector spaces $V^* \to U^*$ in the opposite direction (pull-back of linear forms) and hence an algebra homomorphism $T^*(U) \to T(U^*)$.
We will need mixed tensors which contain co- and contravariant components. We therefore consider

$$T_r^s(U) := \underbrace{U \otimes \cdots \otimes U}_{r} \otimes \underbrace{U^* \otimes \cdots \otimes U^*}_{s} \qquad r, s \in \mathbb{N}_0$$

with the convention $T_0^0(U) := K$, and put

$$T(U) := \bigoplus_{r,s=0}^{\infty} T_r^s(U).$$

Again, the natural bilinear maps

$$\begin{aligned} T_{r_1}^{s_1}(U) \times T_{r_2}^{s_2}(U) &\longrightarrow T_{r_1+r_2}^{s_1+s_2}(U) \\ (u_1 \otimes \cdots \otimes u_{r_1} \otimes u_1^* \otimes \cdots \otimes u_{s_1}^*, v_1 \otimes \dots) &\longrightarrow u_1 \otimes \cdots \otimes v_1 \otimes \cdots \otimes u_1^* \otimes \cdots \otimes v_1^* \otimes \dots \end{aligned}$$

define a product and make $T(U)$ a bigraded associated K-algebra with 1, the <u>tensor algebra</u> of U. We habe natural inclusions $T_*(U) \subset T(U), T^*(U) \subset T(U)$. Elements in $T(U)$ are called <u>tensors</u>, elements in $T_r^s(U)$ are called tensors of type (r,s), e.g. vectors (1,0), linear forms (0,1) , endomorphisms (1,1), scalar products (0,2).
<u>Trace and contraction:</u> From now on, let our "initial" vector spaces $U \dots$ be finite dimensional. Then the natural inclusion $U \hookrightarrow U^{**}$ is an isomorphism and induces natural isomorphisms

$$T_r^s(U^*) \cong T_s^r(U), \quad T(U^*) \cong T(U).$$

The natural bilinear pairing

$$U \times U^* \longrightarrow K, \quad (u, u^*) \longrightarrow u^*(u)$$

induces the linear form

$$T_1^1(U) = U \otimes U^* \cong \mathrm{End}(U) \xrightarrow{\mathrm{tr}} K$$

Indeed, if $\{e_i\}$ is a basis of U and $\{e_i^*\}$ the dual basis of U^*, then the endomorphism $A = \sum_{i,j} a_{ij} \underbrace{e_i \otimes e_j^*}_{\to \delta_{ij}}$ given by the matrix (a_{ij}) is mapped to $\sum_{i,j} a_{ij}\delta_{ij} = \sum_i a_{ii} = \mathrm{tr}(A)$.

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More generally one can pair the i–th covariant component of a homogeneous tensor of type $\geq (i,j)$ with the j–th contravariant component and thus obtain the contraction homomorphisms

$$C_i^j : T_r^s(U) \to T_{r-1}^{s-1}(U)$$

given by

$$u_1 \otimes \cdots \otimes u_r \otimes u_1^* \otimes \cdots \otimes u_s^* \mapsto u_j^*(u_i) \cdot u_1 \otimes \cdots \otimes \widehat{u_i} \otimes \cdots \otimes u_r \otimes u_1^* \otimes \cdots \otimes \widehat{u_j^*} \otimes \cdots \otimes u_s^*.$$

A natural non–degenerate pairing

$$T_r^s(U) \times T_s^r(U) \to K$$

can be obtained e.g. as the composition

$$T_r^s(U) \times T_s^r(U) \to T_r^s(U) \otimes T_s^r(U) \cong T_{r+s}^{r+s}(U) \to K,$$

which is

$$(u_1 \otimes \cdots \otimes u_r \otimes u_1^* \otimes \cdots \otimes u_s^*, v_1 \otimes \cdots \otimes v_s \otimes v_1^* \otimes \cdots \otimes v_r^*) \mapsto \prod_{i=1}^{r} \prod_{j=1}^{s} v_i^*(u_i) u_j^*(v_j).$$

The bases of $T_r^s(U)$ and $T_s^r(U)$ induced by a basis of U are wrt. this pairing dual to each other. The pairing induces a natural isomorphism:

$$(T_r^s(U))^* \cong T_s^r(U) \cong T_r^s(U^*)$$

In particular, we can interprete homogeneous contravariant tensors as multilinear forms and vice versa.

$$\mathrm{Mult}_s(U) \cong (T_s^0(U))^* \cong T_0^s(U) \cong T_s^0(U^*)$$

To the $(0, s)$–tensor $u_1^* \otimes \cdots \otimes u_s^*$ corresponds the multilinear form

$$(u_1, \ldots, u_s) \mapsto \sum_{i=1}^{s} u_i^*(u_i)$$

If $\{e_i\}$ is a basis of U, then

$$\{e_{i_1}^* \otimes \cdots \otimes e_{i_s}^* \mid 1 \leq i_1, \ldots, i_s \leq \dim U\}$$

is a basis of $\mathrm{Mult}_s \cong T_0^s(U)$. A multilinear form $\mu \in \mathrm{Mult}_s(U)$ can be represented wrt. this basis as

$$\mu = \sum_{i_1, \ldots, i_s} \mu(e_{i_1}, \ldots, e_{i_s})\, e_{i_1}^* \otimes \cdots \otimes e_{i_s}^*$$

1.8.2 Exterior product

Suppose now that $\operatorname{char} K = 0$. (This is satisfied for $K = \mathbb{R}, \mathbb{C}$). Let $I \subset T_*(U)$ be the ideal generated by all tensors $u \otimes u$ for $u \in U$. It consists of linear combinations of elements of the form:

$$v_1 \otimes \cdots \otimes v_k \otimes u \otimes u \otimes w_1 \otimes \cdots \otimes w_l \qquad (*)$$

The ideal is graded,

$$I = \bigoplus_{m \geq 0} (I \cap T_m(U))$$

and its homogeneous part of degree m consists of linear combinations of monomials $(*)$ of degree m.

The grading descends to a grading of the quotient algebra

$$\Lambda(U) := T_*(U)/I \cong \bigoplus_{m=0}^{\infty} \underbrace{T_m(U)/I_m}_{=:\Lambda_m}$$

where $\Lambda(U)$ is called the *exterior algebra* (or *Graßmann algebra*) and $\Lambda_m(U)$ the *m–fold exterior power of U*. The identities $\Lambda_0(U) \cong K$ and $\Lambda_1(U) \cong U$ hold due to $I_0 = 0$

and $I_1(U) = 0$. The product induced on $\Lambda(U)$ by $\otimes$ is called *exterior product* or *wedge product*. It is denoted by $\wedge$.

By construction we have the relation $u \wedge u = 0$ for all $u \in U$. This amounts to the anticommutativity

$$u_2 \wedge u_1 = -u_1 \wedge u_2 \qquad \forall u_1, u_2 \in U$$

respectively more generally:

$$u_{\sigma(1)} \wedge \cdots \wedge u_{\sigma(m)} = \operatorname{sgn}(\sigma) u_1 \wedge \cdots \wedge u_m \qquad \forall u_i \in U_i, \sigma \in S_m$$

So

$$b \wedge a = (-1)^{kl} a \wedge b \qquad \forall a \in \Lambda_l(U), b \in \Lambda_k(U)$$

The universal property of the tensor product yields the universal product of the exterior powers: The exterior power translates alternating multilinear maps into linear maps. Equivalently the natrual homomorphism of vector spaces

$$\operatorname{Hom}(\Lambda_m(U), W) \to \operatorname{Mult}_m^{\text{alt}}(U, W)$$

is an isomorphism. In particular $(W = K)$ we have

$$\operatorname{Mult}_m^{\text{alt}}(U) := \operatorname{Mult}_m^{\text{alt}}(U, K) \cong (\Lambda_m(U))^*.$$

We have the natural embedding (which is well–defined due to the universal property of Λ_m):

$$\Lambda_m(U) \hookrightarrow T_m(U), \quad u_1 \wedge \cdots \wedge u_m \mapsto \frac{1}{m!} \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma)\, u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(m)}.$$

It is a right inverse to the canonical projection $T_m(U) \to \Lambda_m(U)$ and therefore injective. To identify multilinear forms with elements in $\Lambda_m(U^*)$, we consider the non–degenerate bilinear pairing:

$$\Lambda_m(U) \times \Lambda_m(U^*) \to K, \quad (u_1 \wedge \cdots \wedge u_m, u_1^* \wedge \cdots \wedge u_m^*) \mapsto \det(u_j^*(u_i)) \qquad (*)$$

It is induced by the analogous pairing

$$T_m(U) \times T_m(U^*) \to K$$

because it maps $I_m(U) \times T_m(U^*) \cup T_m(U) \times I_m(U)$ to zero. The pairing $(*)$ induces the isomorphism of vector spaces:

$$\Lambda_m^*(U) = \Lambda_m(U^*) \to (\Lambda_m(U))^* \cong \operatorname{Mult}_m^{\text{alt}}(U)$$

given by

$$u_1^* \wedge \cdots \wedge u_m^* \mapsto ((u_1, \ldots, u_m) \mapsto \det(u_j^*(u_i))).$$

If $\{e_i\}$ is a basis of U, then $\Lambda_m(U)$ is generated by the elements

$$\{e_{i_1} \wedge \cdots \wedge e_{i_m} \mid 1 \le i_1 \le \cdots \le i_m \le \dim(U)\}.$$

These elements are linearly independent due to

$$(e_{i_1}\wedge\cdots\wedge e_{i_m}, e^*_{j_1}\wedge\cdots\wedge e^*_{j_m})\mapsto \delta_{i_1j_1}\cdots\delta_{i_mj_m}$$

by the above pairing. Hence they form a basis of $\Lambda_m(U)$ and the $e^*_{j_1}\wedge\cdots\wedge e^*_{j_m}$ form the dual basis of $\Lambda_m(U^*)\cong(\Lambda_m(U))^*$. It follows that $\dim\Lambda_m=\binom{\dim U}{m}$. In particular $\Lambda_m(U)=0$ for $m\geq\dim U$ and $\lambda_m(U)\cong K$ (non–canonical) for $m=\dim U$.

For the wedge product of alternating multilinear forms

$$\wedge\colon \operatorname{Mult}^{\text{alt}}_m(U)\times\operatorname{Mult}^{\text{alt}}_n(U)\to\operatorname{Mult}^{\text{alt}}_{m+n}(U)$$

the following formula holds:

$$(\alpha\wedge\beta)(u_1,\dots,u_{m+n})=\frac{1}{m!n!}\sum_{\sigma\in S_{m+n}}\operatorname{sgn}(\sigma)\alpha(u_{\sigma(1)},\dots,u_{\sigma(m)})\beta(u_{\sigma(m+1)},\dots,u_{\sigma(m+n)})$$

It soffices tp cjecl tjos om tje case when α and β are monomials (exercise).

Definition 1.53. Let $m=\dim(U)$. A nontrivial m–multilinear form $0\neq\omega\in\operatorname{Mult}^{\text{alt}}_m(U)\cong\Lambda_m(U^*)\cong K$ is called a *volume form* on U. If $K=\mathbb{R}$, then a component of $\Lambda_m(U^*)\setminus\{0\}$ is an *orientation* of U. Then a basis $\{e_1,\dots,e_m\}$ is *positively* or *negatively oriented* if $\omega(e_1,\dots,e_m)$ for a volume form ω is positive or negative, respectively.

Functoriality: A homomorphism $\alpha\colon U\to V$ of vector spaces induces a homomorphism of vector spces $\Lambda_m(\alpha)\colon\Lambda_m(U)\to\Lambda_m(V)$ and an algebra homomorphism $\Lambda(\alpha)\colon\Lambda(U)\to\Lambda(V)$ given by

$$u_1\wedge\cdots\wedge u_m\mapsto\alpha(u_1)\wedge\cdots\wedge\alpha(u_m).$$

Example. If $A\colon U\to U$ is an endomorphism and $\dim=m$ then $\dim\Lambda_m(U)=1$ and the induced endomorphism $\Lambda_m(A)\colon\Lambda_m(U)\to\Lambda_m(U)$ is the multiplication by a scalar, namely the determinant of A,

$$\Lambda_m(A)=\det(A)\cdot\operatorname{id}_{\Lambda_m(U)}$$

Indeed if A is given wrt. a basis $\{e_i\}$ by a matrix (a_{ij}), then $Ae_j=\sum_i a_{ij}e_i$ and

$$\begin{aligned}\Lambda_m(A)(e_1\wedge\cdots\wedge e_n)&=\left(\sum_{i_1}a_{i_11}e_{i_1}\right)\wedge\cdots\wedge\left(\sum_{i_m}a_{i_mm}e_{i_m}\right)=\\&=\sum_{\sigma\in S_m}\left(\prod_j a_{\sigma(j)j}\right)e_{\sigma(1)}\wedge\cdots\wedge e_{\sigma(m)}=\det(a_{ij})e_1\wedge\cdots\wedge e_m\end{aligned}$$

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1.9 Differential forms and exterior derivative

Differential forms are families of alternating (= skew symmetric) multilinear forms on the tangent spaces of M, i.e. sections of certain vector bundles which we construct first:

$$\Lambda_k^* TM := \bigcup_{p\in M} \underbrace{\Lambda_k^*(T_pM)}_{:=\Lambda_k(T_p^*N)} \xrightarrow{\pi} N$$

in particular

$\Lambda_0^* T_pM \cong \mathbb{R}$ canonically $\rightsquigarrow \Lambda_0^* TM \cong M \times \mathbb{R}$ (trivial line bundle)
$\Lambda_m^* T_pM \cong \mathbb{R}$ non-canonical $\rightsquigarrow$ $\Lambda_m^* TM$ i.g. non-trivial line bundle $\rightsquigarrow$ orientation (later)

Combining these, we obtain the exterior algebra bundle

$$\Lambda^* TM = \bigcup_{p\in M} \underbrace{\underbrace{\Lambda^* T_pM}_{\bigoplus_{k=0}^m \Lambda_k^* T_pM}}_{\text{exterior algebra over } T_p^*M} \xrightarrow{\pi} M$$

Charts (U, x) for M yield charts for $\Lambda_k^* TM$ (resp $\Lambda^* TM$):

$$\Lambda_k^* TU = \bigcup_{p\in U} \Lambda_k^* T_pM \longrightarrow x(U) \times \underbrace{\Lambda_k^* \mathbb{R}^m}_{\cong \mathbb{R}^{\binom{m}{k}}}$$

$$\sum_{1\le i_1 < \ldots < i_k \le m} a_{i_1 \ldots i_k} (dx_{i_1})_p \wedge \ldots \wedge (dx_{i_k})_p \longrightarrow \Big(x(p), \sum a_{i_1 \ldots i_k}(p) e_{i_1}^* \wedge \ldots \wedge e_{i_k}^* \Big)$$

As in the case of TM and T^*M, the coordinate changes are smooth (and defined on open sets) and $\Lambda_k^* TM, \Lambda^* TM$ become $\mathcal{C}^\infty$- manifolds.

Definition 1.54. A smooth differential form of degree k is a smooth section $M \xrightarrow{\alpha} \Lambda_k^* TM$.

A "mixed" differential form is a section $M \longrightarrow \Lambda^* TM$. Hence, if α is a k-form, then $\alpha_p := \alpha(p)$ is an alternating k-multilinear form on T_pM.
In local coordinates

$$\alpha|_U = \sum_{i_1 < \ldots < i_k} a_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

The k-forms form a $\mathcal{C}^\infty(M)$-module $\Omega^k(M)$, the mixed forms a $\mathcal{C}^\infty(M)$-algebra

$$\Omega^*(M) = \bigoplus_{k=0}^m \Omega^k(M)$$

Remark: There is the natural $\mathcal{C}^\infty$-multilin. map, alternating in X_i,

$$\Omega^k(M) \times \Gamma(TM) \times \ldots \times \Gamma(TM) \longrightarrow \mathcal{C}^\infty(M)$$
$$(\alpha, X_1, \ldots, X_k) \longrightarrow \alpha(X_1, \ldots, X_k)$$

and k-fomrs correspond (bijectively) to alternating $\mathcal{C}^\infty(M)$-multilin. maps

$$\Gamma(TM) \times ... \times \Gamma(TM) \longrightarrow \mathcal{C}^\infty(M).$$

The algebra structure on $\Lambda^* T_p M$ depends smoothly on p, thus the pointwise wedge product of smooth forms is again a smooth form

$$\Omega^k(M) \times \Omega^l(M) \xrightarrow[\mathcal{C}^\infty(M)-\text{bilin}]{\wedge} \Omega^{k+l}(M)$$

Pull-back of forms: Let $M \xrightarrow{F} N$ be a smooth map of smooth manifolds. The differentials $T_pM \xrightarrow{dF_p} T_{F(p)}N$ induce degree preserving algebra homom.

$$dF_p^* : \Lambda^* T_{F(p)} N \longrightarrow \Lambda^* T_p M$$

by

$$(dF_p^* \alpha_{F(p)})(v_1, ..., v_k) := \alpha_{F(p)}(dF_p(v_1), ..., dF_p(v_k)),$$

where $\alpha_{F(p)} \in \Lambda_k^* T_{F(p)} N$; $v_1, ... v_k \in T_p M$.
Doing this pointwise yields the degree preserving ($\mathbb{R}$-algebra)-homom

$$F^* : \Omega^*(N) \longrightarrow \Omega^*(M)$$

with $(F^*\alpha)(X_1, ... X_k) = \alpha(dF(X_1), ..., dF(X_k))$, $\alpha \in \Omega^k(M)$; $X_1, ..., X_k \in \Gamma(TM)$.
The smoothness of $F^*\alpha$ is seen in local coordinates:

$$dF \frac{\partial}{\partial x_i}|_p = \sum_j \frac{\partial \tilde{F}_j}{\partial x_i}(x(p)) \frac{\partial}{\partial y_j}|_{F(p)}$$

$$F^*(dy_j)_{F(p)} = \sum_i \underbrace{\left(F^* dy_j \left(\frac{\partial}{\partial x_i}|_p \right) \right)}_{dy_j(dF\frac{\partial}{\partial x_i}|_p) = \frac{\partial \tilde{F}_j}{\partial x_i}(x(p))} (dx_i)_p$$

$$F^* dy_{j_1} \wedge ... \wedge dy_{j_k} = \sum_{i_1, ... i_k} \left(\frac{\partial \tilde{F}_{j_1}}{\partial x_{i_1}} \circ x \right) \cdot ... \cdot \left(\frac{\partial \tilde{F}_{j_k}}{\partial x_{i_k}} \circ x \right) dx_{i_1} \wedge ... \wedge dx_{i_k}$$

$$= \sum_{i_1 < ... < i_k} \underbrace{\sum_{\sigma \in S_k} \left(\frac{\partial \tilde{F}_{ji}}{\partial x_{i_{\sigma(1)}}} \circ x \right) \cdot ... \cdot \left(\frac{\partial \tilde{F}_{jk}}{\partial x_{i_{\sigma(k)}}} \circ x \right) \underbrace{dx_{i_{\sigma(1)}} \wedge ... \wedge dx_{i_{\sigma(k)}}}_{= \text{sgn}(\sigma) dx_{i_1} \wedge ... \wedge dx_{i_k}}}_{= \det \left(\frac{\partial \tilde{F}_{jr}}{\partial x_{i_s}} \circ x \right)_{r,s=1...k} dx_{i_1} \wedge ... \wedge dx_{i_k}}$$

$$F^* \sum_{j_1 < ... < j_k} a_{j_1 ... j_k} dy_{j_1} \wedge ... \wedge dy_{j_k} = \sum_{i_1 < ... < i_k} \underbrace{\left(\sum_{j_1 < ... j_k} a_{j_1 ... j_k} \left(\det \left(\frac{\partial \tilde{F}_{j_r}}{\partial x_{i_s}} \right) \circ x \right) \right)}_{\text{smooth}} dx_{i_1} \wedge ... \wedge dx_{i_k}$$

If $M \subset N$ is a submanifold and $M \xrightarrow{i} N$ the inclusion, then $i^*\alpha$ is called the restriction of α to M.
Exterior derivative: The total differential is a 1. order differential operator

$$d : \mathcal{C}^\infty(M) = \Omega^0(M) \to \Omega^1(M)$$

on functions (= 0-forms), which satisfies the product rule

$$d(fg) = (df)g + f(dg).$$

We extend it to differential forms of degree ≥ 1.

Definition 1.55. An antiderivation of degree $g \in \mathbb{Z}$ on a $\mathbb{Z}-$ graded $\mathbb{R}-$ algebra $A = \bigoplus_{k\in\mathbb{Z}} A_k$ is an $\mathbb{R}-$ linear map $D : A \to A$ with

$$D(A_k) = A_{k+g}$$

satisfying the product rule

$$D(a_k a_l) = (Da_k)a_l + (-1)^k a_k(Da_l), \qquad a_k \in A_k;\ a_l \in A_l$$

Theorem 1.56. *There exists an unique extension of the total differential to an antiderivation of degree +1*

$$d : \Omega^*(M) \to \Omega^*(M) \qquad \underline{\textit{exterior derivative}}$$

with $d^2 = 0$.

Proof. i) Uniqueness: Suppose that d is an exterior derivative. We convince ourselves first that d is a local operator, i.e. that $(d\alpha)_p$ depends only on the germ $[\alpha]_p$. Namely, for $\varphi \in \mathcal{C}^\infty(M)$ holds

$$d(\varphi\alpha) = d\varphi\ \alpha + \varphi d\alpha.$$

We can choose φ s.t. it is supported in a prescribed neighbourhood of p and $\varphi := 1$ near p. Then $d(\varphi\alpha) = d\alpha$ near p.
If $\tilde{\alpha}$ is another form with $[\tilde{\alpha}]_p = [\alpha]_p$, and if we choose φ s.t. $\varphi\tilde{\alpha} = \varphi\alpha$ (everywhere), then

$$d\tilde{\alpha} = d(\varphi\tilde{\alpha}) = d(\varphi\alpha) = d\alpha \text{ near p.}$$

It therefore suffices to check the uniqueness locally in domains of charts.
Let $\alpha \in \Omega^k(M)$. We write α in local coordinates relative to a chart (U,x):

$$\alpha|_U = \sum_{i_1<\ldots<i_k} a_{i_1\ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

The product rule and $d(dx_i) = 0$ imply:

$$d\alpha|_U = \sum da_{i_1\ldots i_k} \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

This determines d.
ii) Existence: On the other hand , we can for a chart (U,x) define an exterior derivative of forms on U by (∗). We verify that it has the required properties. For

$$\alpha = \sum a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k} \in \Omega^k(U) \quad \beta = \sum b_{j_1 \dots j_l} dx_{j_1} \wedge \dots \wedge dx_{j_l} \in \Omega^l(U)$$

holds:

$$\begin{aligned} d^U(\alpha \wedge \beta) &= d^U\Big(\sum_{i_1 < \dots < i_k} \sum_{j_1 < \dots < j_l} a_{i_1 \dots i_k} b_{j_1 \dots j_l} \cdot dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_l} \Big) \\ &= \sum \sum \underbrace{d(a_{i_1 \dots i_k} b_{j_1 \dots j_l})}_{da_{\dots} \cdot b_{\dots} + a_{\dots} \cdot db_{\dots}} \wedge dx_{i_1} \wedge \dots \wedge dx_{j_1} \wedge \dots \wedge dx_{j_l} \\ &= \sum \sum \Big(da_{i_1 \dots i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge b_{j_1 \dots j_l} dx_{j_1} \wedge \dots \wedge dx_{j_l} \\ &+ (-1)^k a_{i_1 \dots i_k} dx_{i_1} \wedge \dots \wedge db_{j_1 \dots j_l} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_l} \Big) \\ &= d^U \alpha \wedge \beta + (-1)^k \alpha \wedge d^U \beta, \end{aligned}$$

i.e. the product rule holds. For $f \in \mathcal{C}^\infty(M)$ holds

$$\begin{aligned} (d^U)^2 f = d^U(\sum_i \frac{\partial f}{\partial x_i} dx_i) &\overset{*}{=} \sum_i \underbrace{d(\frac{\partial f}{\partial x_i})}_{= \sum_j \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j} \wedge dx_i \\ &= \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i = \sum_{i<j} \underbrace{(\frac{\partial^2 f}{\partial x_j \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial x_j})}_{=0} dx_j \wedge dx_i \\ &= 0 \end{aligned}$$

For forms $\alpha \in \Omega^k(U)$ follows

$$(dU)^2 \alpha \overset{(*)^2}{=} \sum \underbrace{(dU)^2 a_{i_1 \dots i_k}}_{=0} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} = 0.$$

Hence d^U is an exterior derivative on U; Since exterior derivatives are unique, d^U cannot depend on the chart (i.e. on the coo. x) and we obtain well-defined exterior derivatives on all chart domains (i.e. open sets diffeom to open subsets of eucl. space).
If $W \subset U$ is open, (U,x) a chart, we can restrict the chart to W, and have

$$d^W(\alpha|_W) = (d^U \alpha)|_W \qquad \alpha \in \Omega^*(U)$$

Therefore, if (U', x') is another chart, then for $\alpha \in \Omega^*(M)$ holds:

$$(d^U(\alpha|_U))|_{U \cap U'} = d^{U \cap U'} \alpha|_{U \cap U'} = (d^{U'}(\alpha|_{U'}))|_{U \cap U'}$$

The exterior derivatives on the various chart domains are thus compatible and can be put together to define a global exterior derivative:

$$d\alpha|_U := d^U(\alpha|_U) \qquad \alpha \in \Omega^*(M)$$

□

Lemma 1.57. *(compatible with pull-back:) For $M \xrightarrow{F} N$ smooth holds*

$$d^M \circ F^* = F^* \circ d^N$$

i.e.

$$d^M(F^*\alpha) = F^*(d^N\alpha) \qquad \alpha \in \Omega^*(N)$$

08.01.2012

1.10 Partitions of unity

A partition of unity is used to globalise local constructions on manifolds. Let M be a smooth manifold. We will use the existence of 'bump functions': For $p \in M$ and a neighborhood U of this point there exists a smooth function $\varphi\colon M \to [0,1]$ with compact support $\operatorname{supp}(\varphi) \subset U$ and $\phi \equiv 1$ near p.

Definition 1.58. A smooth *partition of unity* on M is a family $(\varphi_\iota)_{\iota\in I}$ of smooth functions $\varphi_\iota\colon M \to [0,1]$ such that the following holds:

1. The family of supports $\operatorname{supp}(\varphi_\iota)$ is *locally finite*, i.e. every point has a neighborhood which intersects only finitely many supports $\operatorname{supp}(\varphi_\iota)$. As a consequence, any compact subset intersects only finitely many of these supports.

2. The functions sum up to unity
$$\sum_{\iota\in I} \varphi_\iota \equiv 1$$

The partition of unity is called *subordinate* to an open covering $(U_\alpha)_{\alpha\in A}$ of M, if there exists a map of index sets $\widetilde{\alpha}\colon I \to A$ such that $\operatorname{supp}(\varphi_\iota) \subset U_{\widetilde{\alpha}(\iota)}$ for all $\iota \in I$.

Theorem 1.59. *For every open covering $(U_\alpha)_{\alpha\in A}$ of M there exists a partition of unity $(\varphi_\iota)_{\iota\in I}$ subordinate to the covering with compact supports* $\operatorname{supp}(\varphi_\iota)$ *and a countable index set I.*

Lemma 1.60. *M admits a compact exhaustion, i.e. there exists an ascending sequence of open relatively compact subsets G_n, $G_1 \subset G_2 \subset \dots$, such that $\overline{G_n} \subset G_{n+1}$ for all $n \in \mathbb{N}$ and $\bigcup G_n = M$.*

Proof. The topology of M has a countable basis consisting of relatively compact open subsets B_i, $i \in \mathbb{N}$, since the relatively compact subsets in a given basis form a basis themselves. We inductively construct a sequence $(i_n)_{n\in\mathbb{N}}$. Choose $i_1 = 1$, then suppose i_n has already been constructed, then $G_n \coloneqq \bigcup_{i=1}^{i_n} B_i$ is relatively compact. We choose $i_{n+1} > i_n + 1$ as the minimal number such that $\overline{G_n} \subset \bigcup_{i=1}^{i_{n+1}} B_i$. The sets G_n have the required properties. □

Proof of the theorem. For every point $p \in \overline{G_n} \setminus G_{n-1}$ (compact) we choose an open neighborhood $V_p \subset \overline{G_{n+1}} \setminus G_{n-2}$ and a smooth function $\psi_p\colon M \to [0,1]$ with $\operatorname{supp}(\psi_p) \subset V_p$ with $\psi_p(p) > 0$. By compactness, finitely many points p suffice to cover $\overline{G_n} \setminus G_{n-1}$ by the corresponding open subsets $\{\psi_p > 0\}$. Hence there exists a countable set $\{p_\iota \mid \iota \in I\}$ of points such that

- The set $\{\psi_{p_\iota} > 0\}$ cover M.
- The family of sets V_{p_ι} is locally finite the function $\psi := \sum \psi_{p_\iota}$ is then well–defined, smooth and strictly positive.

The normalised functions $\phi_\iota := \psi_{p_\iota}/\psi$ form a partition of unity with compact supports. In order to adopt the partitions of unity to a given covering $(U_\alpha)_{\alpha\in A}$ we choose every neighborhood V_p such that it is contained is one of the covering sets $U_{\widetilde{\alpha(p)}}$. □

1.11 Orientations

Let V be a d–dimensional $\mathbb{R}$–vector space. Recall from linear algebra that $\mathrm{GL}(V)$ has two connected components, where $\mathrm{GL}^+(V)$ denotes the component containing the identity. Since $\mathrm{GL}(V)$ operates simply transitively on the set of bases of V, the group $GL^+(V)$ has two orbits. An *orientation* of V is a choice of one of these orbits. The basis in this orbit are then called *positively oriented*, the others are *negatively oriented*. Thus the orientation is a continuous map from the d^2–dimensional space of bases of V to $\{-1, 1\}$.

Two bases $\{e_i\}$,$\{\widetilde{e_j}\}$ of V have the same orientation if $e_i = \sum_j a_{ij}\widetilde{e_j}$ with $\det(a_{ij}) > 0$. A top–dimensional form $\omega \in \Lambda_d(V^*) \cong \mathbb{R}$ determines an orientation by defining a basis $(e_1, \ldots, e_d)$ as positively oriented, if $\omega(e_1, \ldots, e_d) > 0$.

In the case $\dim V = 0$ one makes the convention that the orientation consists in the choice of a sign $\pm$. A 0–dimensional vector space has a natural orientation, namely $+$.

Remark. A finite dimensional complex vector space has a preferred orientation as a real vector space, because the map

$$\{\mathbb{C}\text{–basis}\} \to \{\mathbb{R}\text{–basis}\}, \quad (e_1, \ldots, e_j) \mapsto (e_1, ie_1, \ldots, e_n, ie_n)$$

distinguishes an orientation of $\mathbb{R}$–bases.

Let now M be a smooth manifold (C^1 suffices). An orientation of M is, intuitively speaking, a family of orientations of the tangent spaces T_pM which varies continously on p. This can be made precise as follows:

Definition 1.61. A *local basefield* is an m–tuple (E_i) of vector fields $E_1, \ldots, E_m \in \Gamma(TM)$ defined on an open subset $U \subset M$ such that $(E_i(p))$ is a basis for T_pM for all $p \in U$. For instance, local coordintes (U, x) yield the smooth local base field $(\partial/\partial x_i)$ on U. An *orientation* of M is a simultaneous orientation of all tangent spaces T_pM such that for every (continuous) local base field (E_i) the orientation sign of $(E_i(p))$ is locally constant (as a function of p). M is *orientable*, if an orientation exists. In dimension 0, one makes the convention that the orientation of a point consists in the choice of a sign $\pm$.

Definition 1.62. A *volume form* on an m–dimensional manifold M is a nowhere vanishing top dimensional form $\omega \in \Omega^m(M)$, i.e. $\omega_p \neq 0$ for all $\in M$. Volume forms induce orientations: Call a basis $(v_1, \ldots, v_m)$ of T_pM positively oriented if $\omega_p(v_1, \ldots, v_m) > 0$.

Proposition 1.63. *Every orientation comes from a volume form.*

Proof. If M is oriented, we can cover M by coordinate charts (U, x) such that the local basefields $(\partial/\partial x_i)$ are positively oriented. The orientation is induced on U by the volume form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$. These local volume forms are positive multiples of each other on the regions of overlap. They can be combined to a global volume form inducing the given orientation using a partition of unity. □

Remark. If ω_1 and ω_2 are volume fomres, then there exists a smooth function $f\colon M \to \mathbb{R}^\times$ with $\omega_2 = f\omega_1$. Both volume fomrs induce the same orientation if and only if $f > 0$. If M is connected and orientable, then M has exactly two orientations.

Example. The standard orientation of $\mathbb{R}^m$ is defined by choosing the global base field $(\partial/\partial x_i)$ as positively oriented and it is induces by $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$.

Definition 1.64. A local diffeomorphism $F\colon M \to N$ of oriented manifolds is *orientation preserving* if its differential maps positively oriented bases to positively oriented bases.

Orientation preserving diffeomorphisms F between open subsets of $\mathbb{R}^d$ are distinguished by the fact that the Jacobian determinant $\det(\partial F_i/\partial x_j)$ is positive.

For smooth maniolds M one can consider the subatlas $\mathcal{A}_+$ of the differentiable structur which consists of all orientation–preserving charts (where $\mathbb{R}^m$ is equipped with its standard orientation). Vice–versa, a smooth atlas with orientation preserving coordinate changes defines an oriented smooth structure.

10.01.2012

1.12 Manifolds with boudary

The local model for a d-dim manifold with boundary, $d \geq 1$, is $H^d := \{x_1 \leq 0\} \subset \mathbb{R}^d$. Partial derivatives and total differential of functions $H^d \xrightarrow{f} \mathbb{R}^d$ are also defined in boundary points by linear approximation. It is technically convenient to define tangent vectors in boundary points as derivations. Then for all points $p \in H^d$, whether they are interior or boundary points, holds

$$T_p H^d \cong \mathbb{R}^d \qquad \text{canonically}$$

Definition 1.65. A d-dim locally euclidean space M^d with boundary, $d \geq 1$, is a topological space locally homeomorphic to H^d. If in addition m is Hausdorff and second countable, then M is a d-dim topological manifold with boundary

To define differentiable structures we use as charts homeomorphisms $U \xrightarrow{x} x(U)$ from open subsets $U \subset M$ onto open subsets of H^d. A smooth structure is again defined as an atlas with smooth coordinate changes. If M^d is a smooth manifold with boundary, then $p \in M$ is an interior resp. boundary point if it is mapped by a chart to an interior, resp. boundary point of H^d. This is independent of the chart because coordinate changes map neighbourhoods of interior points to open subsets of $\mathbb{R}^d$ and hence interior points to interior points.

The interior of M is an open subset and inherits a structure as d-dim manifold without boundary. The boundary ∂M of M is closed and inherits a structure as $(d-1)$-dim manifold without boundary.
Example:

1) $[a, b]$ 1-manifold with boundary $\{a, b\}$

2) If $M^{d\geq 1}$ is a smooth manifold without boundary, $f \in \mathcal{C}^\infty(M)$ and $a \in \mathbb{R}$ is a regular value of f, then the sublevel $\{f \leq a\}$ is a smooth manifold with boundary $\{f = a\}$. E.g. $M = \mathbb{R}^d$, $f(x) = \|x\|^2 \Rightarrow \{f \leq 1\} = B_1^d(0)$ (unit ball with boundary the unit sphere.)

Let $M^{d\geq 2}$ be a smooth manifold with boundary. Tf M is oriented, then ∂M is orientable and we choose an orientation of the boundary by the following construction: Let $\mathcal{A}_+$ be the subatlas of the smooth structures of M consisting of all orientation preserving charts ($\mathbb{R}^d$ equipped with standard orientation). The restrictions of these charts to ∂M yield an atlas for ∂M with orientation preserving coordinate changes. If one equips $\partial H^d \cong \mathbb{R}^{d-1}$ with the orientation such that the base field $(\frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d})$ is positively oriented, then this determines an orientation of ∂M.
It can also be describes as follows: A tangent vector $v \in T_pM$ in $p \in \partial M$ points to the outside if with respect to some (any) chart its $\frac{\partial}{\partial x_1}$-compontent is positive. A basis $(v_2, \dots v_d)$ of $T_p\partial M$ is positively oriented if and only if the basis $(v, v_2, \dots v_d)$ of T_pM is positively oriented. In the case dim M = 1, ∂M is descrete and we give $p \in \partial M$ the positive orientation if tangent vectors in p pointing to the outside are positively oriented.

1.13 Integration of differential forms over manifolds

M^m smooth, oriented, with boundary (possibly empty). One denotes by $\Omega^*_{\text{cpt}}(M)$ the $\mathcal{C}^\infty(M)$-module of smooth forms with compact support. We want to define the integral $\int_M \alpha$ for $\alpha \in \Omega^m_{\text{cpt}}(M)$.
Preparation: $U \subset H$ open with standard orientation $\alpha = f\ dx_1 \wedge \dots \wedge dx_m$
$f \in \mathcal{C}^\infty_{\text{cpt}}(U)$. We put

$$\int d\alpha := \int f\ dx_1 \dots dx_m.$$

Lemma 1.66. *If $U \xrightarrow{F} V$ is an orientation preserving diffeo of open subsets of H^m and $\alpha \in \Omega^m_{cpt}(V)$, then*

$$\int\limits_U F^*\alpha = \int\limits_V \alpha$$

Proof. We write $\alpha = f\ dy_1 \wedge \dots \wedge dy_m$. In order to determine $F^*_\alpha = ?\ dx_1 \wedge \dots \wedge dx_m$

we compute:

$$(F^*\alpha)\Big(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_m}\Big) = \alpha\Big(\underbrace{dF\frac{\partial}{\partial x_1}}_{=\sum_j \frac{\partial F_j}{\partial x_1}(\frac{\partial}{\partial y_j}\circ F)},\dots,dF\frac{\partial}{\partial x_m}\Big)$$

$$= \det\Big(\frac{\partial F_j}{\partial x_i}\Big)\cdot \underbrace{\alpha\Big(\frac{\partial}{\partial y_1}\circ F,\dots,\frac{\partial}{\partial y_m}\circ F\Big)}_{=(f\circ F)\underbrace{(dy_1\wedge\dots\wedge dy_m)\Big(\frac{\partial}{\partial y_1}\circ F,\dots,\frac{\partial}{\partial y_m}\circ F\Big)}_{=1}}$$

$\Rightarrow F^*\alpha = (f\circ F)\cdot\det(\frac{\partial F_i}{\partial y_j})\cdot dx_1\wedge\dots\wedge dx_m$
Applying the transformation formula for integrals, we obtain:

$$\int\limits_V \alpha = \int\limits_V f dy_1\dots dy_m = \int\limits_U (f\circ F)|\underbrace{\det(\frac{\partial F_i}{\partial y_j})}_{>0,\text{ b/c F orient. pres.}}|dx_1\dots dx_m$$

$$= \int\limits_U (f\circ F)\det(\frac{\partial F_i}{\partial x_j})dx_1\dots dx_m = \int\limits_U F^*\alpha$$

□

If $\alpha\in\Omega^m_{cpt}(M)$ has support in a chart domain U and if $U\xrightarrow{x}x(U)$ is an orientation preserving chart, then we define

$$\int\limits_M \alpha := \int\limits_{x(U)} (x^{-1})^*\alpha.$$

Due to the lemma, the integral is independent of the chart.
In order to define the integral for an arbitrary form $\alpha\in\Omega^m_{cpt}(M)$ we consider a covering $(U_\beta)_{\beta\in B}$ of M by charts and a partition of unity $(\varphi_i)_{i\in I}$ subordinate to it, $\sum_i\varphi_i := 1$, and we put

$$\int\limits_M \alpha := \sum_{i\in I}\int\limits_M \varphi_i\ \alpha.$$

Note that this is a finite sum because supp(α) is compact and the partition of unity locally finite.
If $(V_j)_{j\in J}$ is another such covering and $(\Psi_j)_{j\in J}$ a partition of unity subordinate to it, then

$$\sum_i \underbrace{\int\limits_M \varphi_i\ \alpha}_{\sum_j\int_M \varphi_i\ \Psi_j\ \alpha} = \sum_{i,j}\int\limits_M \varphi_i\ \Psi_j\ \alpha = \sum_j\int\limits_M \Psi_j\ \alpha.$$

Hence the integral is well defined.
The lemma above generalizes:
Diffeomorphism invariance: If $M^d \xrightarrow{F} N^d$ is an orientation preserving diffeomorphism of oriented manifolds, then holds for $\alpha \in \Omega^d_{\text{cpt}}(N)$:

$$\int_M F^*\alpha = \int_N \alpha$$

1.14 Stokes' Theorem

Theorem 1.67. *Let $M^{m\geq 1}$ be an oriented smooth manifold with boundary and $\alpha \in \Omega^{m-1}_{\text{cpt}}(M)$, then*

$$\int_M d\alpha = \int_{\partial M} \alpha,$$

where the last integral means integration of $\alpha|_{\partial M}$ using the induced orientation of ∂M.

Remark. Let $M = [a,b]$ with the standard orientation induced by $\mathrm{d}x$ and $f \in \mathscr{C}^\infty(M) = \Omega^0(M)$. Then

$$\int_{[a,b]} \mathrm{d}f = \int_{[a,b]} f' \,\mathrm{d}x = \int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a) = \int_{\partial[a,b]} f.$$

So in one dimension, Stokes reduces to the fundamental theorem of calculus.

15.01.2012

Proof. Let first $M = H^m = \{x_1 \leq 0\} \subset \mathbb{R}^m$ equipped with the standard orientation by $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m$. The induced orientation on the boundary $\partial H^n = \{x_0 = 0\}$ is given by $\mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_m$. We write

$$\alpha := \sum_{i=1}^n \alpha_i := \sum_{i=1}^m f_i \,\mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_m \qquad \mathrm{d}\alpha_i = \frac{\partial f_i}{\partial x_i}(-1)^{i-1}\,\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_m.$$

With Fubini and the fundamental theorem of calculus we obtain

$$\begin{aligned}\int_{H^m} \mathrm{d}\alpha_1 &= \int_{H^m} \frac{\partial f_i}{\partial x_i}\,\mathrm{d}x_1 \cdots \mathrm{d}x_m = \int_{H^m} \left(\int_{-\infty}^0 \frac{\partial f_i}{\partial x_i}(x_1,\dots,x_m)\,\mathrm{d}x_1\right)\mathrm{d}x_2 \cdots \mathrm{d}x_m \\ &= \int_{\partial H^m} f_1 \,\mathrm{d}x_2 \cdots x_m = \int_{\partial H^m} \alpha_1.\end{aligned}$$

For $i > 1$ one obtains similarly

$$\int_{H^m} \mathrm{d}\alpha_i = (-1)^{i-1} \int_{H^{m-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f_i}{\partial x_i}(x_1,\dots,x_m)\,\mathrm{d}x_i\right)\mathrm{d}x_1 \cdots \widehat{\mathrm{d}x_i} \cdots \mathrm{d}x_m = 0 = \int_{\partial H^m} \alpha_i$$

since $\alpha_i|_{\partial H^m} = 0$. So the statement of Stokes' theorem holds for $M = H^m$. The general case follows: We choose a partition of unity (φ_ι) which is subordinate to an open cover by charts. Since Stokes holds for forms which are compactly supproted in charts,

$$\int \mathrm{d}(\varphi_\iota \alpha) = \int_{\partial M} \varphi_\iota \alpha.$$

The assertion follows by summation (both sums are finite since the support of α is compact). $\square$

1.15 The Poincaré lemma

Definition 1.68. A differential form α is called *closed* if $\mathrm{d}\alpha = 0$. It is called *exact* if there exists a form β such that $\alpha = \mathrm{d}\beta$.

Since $\mathrm{d}^2 = 0$, exact forms are always closed. As an a posteriori motivation for the introduction of higher differential forms and the exterior derivative we now show (improving the earlier discussion for 1–forms) that closedness is the precise local obstruction to exactness.

17.01.2012

1.16 Cohomology

The sequence of vector space (in fact $\mathcal{C}^\infty(M)$–module) homomorphisms

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} ... \quad \text{(terminates after finitely many steps)}$$

with $d^2 = 0$ (complex property $\rightsquigarrow$ homological alg.) is called the *de Rham complex* of M. The complex property is equivalent to the fact that exact forms are closed.

The discrepancy between closed and exact forms is measured by the de Rham cohomology groups (which are in fact vector spaces)

$$H^k_{(\text{de Rham})}(M) := \frac{\ker d|_{\Omega^k(M)}}{\operatorname{im} d|_{\Omega^{k-1}(M)}} \qquad k \geq 0$$

They are 'global obstructions to exactness'.

The wedge product of forms induces a multiplicative structure on

$$H^*(M) := \bigoplus_{k \geq 0} H^k(M),$$

namely the cup product:

$$\begin{aligned} H^k(M) \times H^l(M) &\xrightarrow[\text{bilinear}]{\cup} H^{k+l}(M) \\ ([\alpha],[\beta]) &\longrightarrow [\alpha \wedge \beta] =: [\alpha] \cup [\beta] \end{aligned}$$

The cup product is well defined, because the wedge product of a closed form and an exact form is exact:

$$d\alpha = 0,\ \beta = d\gamma \quad \Rightarrow \quad \alpha \wedge \beta = \pm \underbrace{d(\alpha \wedge \beta)}_{\underbrace{d\alpha}_{=0} \wedge \gamma \pm \alpha \wedge \underbrace{d\gamma}_{=\beta}}$$

$H^*(M)$ thus becomes an anticommutative, graded, associative $\mathbb{R}$–algebra with unity (represented by the constant function 1), the *de Rham cohomology ring*.

$$1_{H^*} = [1] \in H^0$$

A smooth map $F\colon M \to N$ induces a *homomorphism of complexes*, i.e. the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega^0(N) & \xrightarrow{\mathrm{d}} & \cdots & \xrightarrow{\mathrm{d}} & \Omega^k(N) & \xrightarrow{\mathrm{d}} & \cdots \\ & & \big\downarrow{\scriptstyle F^*} & & & & \big\downarrow{\scriptstyle F^*} & & \\ 0 & \longrightarrow & \Omega^0(M) & \xrightarrow{\mathrm{d}} & \cdots & \xrightarrow{\mathrm{d}} & \Omega^k(M) & \xrightarrow{\mathrm{d}} & \cdots \end{array}$$

commutes, where F^* is the pullback of differential forms. The pullback induces a map on the cohomology groups and this forms a homomorphism of graded algebras:

$$\begin{aligned} H^*(N) &\xrightarrow{F^*} H^*(M) \\ [\alpha] &\longmapsto [F^*\alpha] \end{aligned}$$

In particular, de Rham cohomology is a differentiable invariant, i.e. diffeomorphisms of manifolds induce isomorphisms on cohomology.

The 0^{th} cohomology group is the set of locally constant smooth functions on M, so it is isomorphic to $\mathbb{R}^{\pi_0(M)}$, where $\pi_0(M)$ denotes the set of connected components of M.

$$H^0(M) \cong \{f \in \underbrace{\mathcal{C}^\infty(M)}_{\Omega^0(M)} : \underbrace{f \text{ locally const.}}_{\Leftrightarrow df=0 \text{ “closed”}}\} \cong \underbrace{\mathbb{R}^{\pi_0(M)}}_{\{\text{maps } \pi_0(M)\to\mathbb{R}\}}$$

The higher cohomology groups contain nontrivial information:

1) Consider $S^1 \subset \mathbb{C}$, $d\theta \in \Omega^1(S^1)$. The function θ is only well defined up to an additive constant in $2\pi\mathbb{Z}$. It is locally equal to $\arg(x+iy) = \arctan\frac{y}{x} = \frac{\pi}{2} + \arctan(\frac{-x}{y})$. Since θ is not globally defined, $\mathrm{d}\theta$ is in fact not exact.

$$\int\limits_{S^1} d\theta = 2\pi \overset{\text{Stokes}}{\Longrightarrow} d\theta|_{S^1} \text{ is not exact.}$$

   Hence $[d\theta] \neq 0$ in $H^1(S^1)$. More precisely, for $\alpha \in \Omega^1(S^1)$ we have

$$\int\limits_{S^1} \alpha = 0 \Longleftrightarrow \alpha \text{ exact}$$

   So integration over S^1 gives an isomorphism $H^1(S^1) \to \mathbb{R}$.

2) Let M^m be a closed (i.e. compact without boundary), orientable manifold and $\omega \in \Omega^m(M)$ volume form (i.e. nowhere vanishing). Choose an orientation on M compatible with ω, so $\int\limits_M \omega > 0$. Then by Stokes' theorem, $[\omega] \neq 0$ in $H^m(M)$. More precisely, one can prove, that

$$H^m(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega,$$

   which is well–defined by Stokes, is fact an isomorphism.

3) left out

4) Let $N^k \subset M^m$ be a submanifold. Assume N is a *retract* of M, i.e. there exists a smooth *retraction* $r\colon M \to N$, such that $r \circ \iota = \mathrm{id}_N$, where $\iota\colon N \to M$ is the inclusion map. Then by the functoriality of H^*, the induced homomorphisms $r^*\colon H^*(N) \to H^*(M)$ and $\iota^*\colon H^*(M) \to H^*(N)$ satisfy $\iota^* \circ r^* = \mathrm{id}_{H^*(N)}$, so r^* is injective. So the nontrivial cohomology of N can also be found in M.

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If $F\colon M \to N$ is a diffeomorphism, then the induced homomorphism F^* on cohomology is an isomorphism.

Example.

1. S^{n-1} is a retract of $\mathbb{R}^n \setminus \{0\}$, so $H^{n-1}(\mathbb{R}^n \setminus \{0\}) \neq 0$.
2. $\mathbb{R}^3 \setminus \{x, y = 0\}$ retracts to a (suitably embedded) copy of S^1, so $H^1(\mathbb{R}^3 \setminus \{\text{line}\}) \neq 0$. (In fact, it is isomorphic to $\mathbb{R}$).
3. Consider a projection $p_1\colon M_1 \times M_2 \to M_1$.

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1.17 Vector calculus on $\mathbb{R}^3$

On $\mathbb{R}^3$, we have the standard scalar product

$$\langle -, - \rangle = \mathrm{d}x_1 \otimes \mathrm{d}x_1 + \mathrm{d}x_2 \otimes \mathrm{d}x_2 + \mathrm{d}x_3 \otimes \mathrm{d}x_3,$$

which induces the standard volume form $\mathrm{d}V = \mathrm{vol} = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3$.

We have from the exercises:

$$\begin{array}{ccccccc}
\Omega^0(\mathbb{R}^3) & \xrightarrow{\mathrm{d}} & \Omega^1(\mathbb{R}^3) & \xrightarrow{\mathrm{d}} & \Omega^2(\mathbb{R}^3) & \xrightarrow{\mathrm{d}} & \Omega^3(\mathbb{R}^3) \\
\uparrow \iota_0 & & \uparrow \iota_1 & & \uparrow \iota_2 & & \uparrow \iota_3 \\
\mathscr{C}^\infty(\mathbb{R}^3) & \xrightarrow{\mathrm{grad}} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\mathrm{rot}} & \Gamma(T\mathbb{R}^3) & \xrightarrow{\mathrm{div}} & \mathscr{C}^\infty(\mathbb{R}^3)
\end{array}$$

We can integrate forms over oriented submanifolds. The metric (inherited from $\mathbb{R}^3$) and orientation (inherited by domains, to be chosen for curves and surfaces) determine distinguished volume forms on the submanifolds, which allow us to measure length, area and volume.

In case of a 3–dimensional submanifold $\Omega^3 \subset \mathbb{R}^3$, we use the metric and orientation of $\mathbb{R}^3$ and get the volume form

$$\mathrm{d}V = \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}x_3.$$

If we have a 2–dimensional submanifold $\Sigma^2 \subset \mathbb{R}^3$, there is an induced metric from $\mathbb{R}^3$ and we have to choose an orienation by giving a unit normal vector field ν along Σ.

Then (v_1, v_2) is defined to be a positively oriented basis of $T_x\Sigma$ if and only if (ν, v_1, v_2) is a positively oriented basis of $\mathbb{R}^3$. The 2–dimensional volume form on Σ, the *area element* is then given by

$$\mathrm{d}A = \mathrm{d}V(\nu, -, -)|_\Sigma \in \Omega^2(\Sigma).$$

A pair (v_1, v_2) is a positively oriented orthonormal basis of $T_x\Sigma$ if and only if $\mathrm{d}A_x(v_1, v_2) = +1$.

Now let $C^1 \subset \mathbb{R}^3$ be a curve, i.e. a 1–dimensional submanifold. Choosing a unit tangent vector field T along C defines an orientation on C, and the 1–dimensional volume form, the *length element*, is then given by

$$\mathrm{d}L = \langle T, - \rangle.$$

We now rewrite integrals of forms over submanifolds and interpret them physically/geometrically: Let $\Omega^3, \Sigma^2, C^1 \subset \mathbb{R}^3$ be 3,2,1–dimensinal submanifolds of $\mathbb{R}^3$. Then

$$\begin{aligned} \iota_3 f = f\,\mathrm{d}V \quad &\Rightarrow \quad \int_\Omega \iota_3 f = \int_\Omega f\,\mathrm{d}V \\ \iota_2 X|_\Sigma = \mathrm{d}V(X, -, -)|_\Sigma = \langle X, \nu\rangle\,\mathrm{d}V(\nu, -, -)|_\Sigma \quad &\Rightarrow \quad \int_\Sigma \iota_2 X = \int_\Sigma \langle X, \nu\rangle\,\mathrm{d}A \\ \iota_1 X|_C = \langle X, -\rangle|_C = \langle X, T\rangle\,\mathrm{d}L \quad &\Rightarrow \quad \int_C \iota_1 X = \int_C \langle X, T\rangle\,\mathrm{d}L \end{aligned}$$

Now we can translate Stokes' theorem into classical integral theorems: For a compactly supported vector field X along Σ, we get

$$\int_\Sigma \langle \operatorname{rot} X, \nu\rangle\,\mathrm{d}A = \int_\Sigma (\iota_2 \operatorname{rot} X)|_\Sigma = \int_\Sigma \mathrm{d}(\iota_1 X) = \int_{\partial\Sigma} \iota_1 X = \int_{\partial\Sigma} \langle X, T\rangle\,\mathrm{d}L,$$

which is exactly the classical *Kelvin–Stokes theorem.* Now let X be a compactly supported vector field on Ω. Then

$$\int_\Omega \operatorname{div} X\,\mathrm{d}V = \int_\Omega \iota_3 \operatorname{div} X = \int_\Omega \mathrm{d}(\iota_2 X) = \int_{\partial\Omega} \langle X, \nu\rangle\,\mathrm{d}A,$$

which is the *Gauß integral theorem.* As an application of the Gauß theorem, we insert $X = g \cdot \operatorname{grad} f$. Then $\operatorname{div} X = \langle \operatorname{grad} g, \operatorname{grad} f\rangle + g\Delta f$, so we get

$$\int_\Omega (\langle \operatorname{grad} g, \operatorname{grad} f\rangle + g\Delta f)\,\mathrm{d}V = \int_{\partial\Omega} g \cdot \partial_\nu f\,\mathrm{d}A,$$

which is called *Green's formula.* Of particular importance is the case $g = 1$, where the formula reduces to

$$\int_\Omega \Delta f\,\mathrm{d}V = \int_{\partial\Omega} \partial_\nu f\,\mathrm{d}A.$$

Using the classical integral theorems, we can (by integration) interpret the differential operators div and rot physically/geometrically:

1. We apply the Stokes integral theorem to small flat disks ($r \approx 0$) with unit normal vector field ν:
$$\int_{D_r} \langle \operatorname{rot} X, \nu \rangle \, \mathrm{d}A = \int_{\partial D_r} \langle X, T \rangle \, \mathrm{d}L$$
The quantity $\langle \operatorname{rot} X, \nu \rangle$ can then be interpreted as the circulation density of X per area orthogonal to ν, and $\operatorname{rot} X$ therefore is the vectorial *circulation density* per area. The statement of Stokes' theorem is then that the summed circulation density of X over Σ equals the circulation of X along $\partial\Sigma$.

2. We apply the Gauß integral theorem to small balls:
$$\int_{B_r} \operatorname{div} X \, \mathrm{d}V = \int_{\partial B_r} \langle X, \nu \rangle \, \mathrm{d}A$$
This leads to the interpretation of $\operatorname{div} X$ as the source density of X. The Gauß theorem states that the integrated *source density* of Ω equals the outward flux through $\partial\Omega$.

Example (Electrodynamics)**.** We first work on $\mathbb{R}^3$. The electric field is a time–dependent vector field $E = E_x\partial_x + E_y\partial_y + E_z\partial_z$. We consider it as a 1–form

$$\iota_1 E = E_x \, \mathrm{d}x + E_y \, \mathrm{d}y + E_z \, \mathrm{d}z$$

It is measured by its force on test charges, which is independent of the velocity. The magnetic field B is also a time dependent vector field. It is measured by its Lorentz force on moving charges, which does depend on their velocity ($L = v \times B$). This makes it very natural to consider it as a 2–form $\iota_2 B = \operatorname{vol}(B, -, -)$. We also have the electric charge dencity ρ, which is a time dependent function considered as a 3–form $\iota_3\rho = \rho \, \mathrm{d}V$ and the electric current density J, which is to be considered as a 2–form $\iota_2 J$.

Electric and magnetic fields, their interaction and generation by charges and currents, are described by Maxwell's equations. They can be formulated as a system of PDE's, which is the most efficient description for computations, or alternatively in an integrated form, which reflects empirical evidence more directly and thus allows physical interpretation.

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